Universität Koblenz-Landau

FB 4 Informatik

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Exercises for "Decision Procedures for Verification"
Exercise sheet 5

Exercise 5.1: *(4 P)*
Prove or refute the following statements:

(a) If F is a first-order formula, then F is valid if and only if $F \to \bot$ is unsatisfiable.

(b) If F and G are first-order formulae, F is valid, and $F \to G$ is valid, then G is valid.

(c) If F and G are first-order formulae, F is satisfiable, and $F \to G$ is satisfiable, then G is satisfiable.

(d) If F is a first-order formula and x a variable, then F is unsatisfiable if and only if $\exists x F$ is unsatisfiable.

(e) If F and G are first-order formulae and x is a variable then $\forall x(F \wedge G) \models \forall x F \wedge \forall x G$ and $\forall x F \wedge \forall x G \models \forall x(F \wedge G)$.

(f) If F and G are first-order formulae and x is a variable then $\exists x(F \wedge G) \models \exists x F \wedge \exists x G$ and $\exists x F \wedge \exists x G \models \exists x(F \wedge G)$.

Exercise 5.2: *(3 P)*
Let $\Sigma = (\Omega, \Pi)$, where $\Omega = \{0/0, s/1, +/2\}$ and $\Pi = \emptyset$ (i.e. the only predicate symbol is $\approx$). Consider the following formulae in the signature Σ:

1. $F_1 = \forall x \ (x + 0 \approx x)$
2. $F_2 = \forall x, y \ (x + s(y) \approx s(x + y))$
3. $F_3 = \forall x, y \ \ (x + y \approx y + x)$.

Find a Σ-structure in which F_1 and F_2 are valid but F_3 is not.

Exercise 5.3: *(3 P)*
What is the clausal normal form of

$$\exists x \, \forall y \, (\forall z \, (p(y, z) \vee \neg \, x \approx y) \to (\forall z \, q(y, z) \wedge \neg \, r(x, y)))$$

Supplementary exercise.

Exercise 5.4: *(3 P)*
Let F be a closed first-order formula with equality over a signature $\Sigma = (\Omega, \Pi)$. Let $\sim \notin \Omega$ be a new binary relation symbol (written as an infix operator). Let the set $Eq(\Sigma)$ contain the formulas

$$\forall x\,(x \sim x)$$
$$\forall x, y\,(x \sim y \rightarrow y \sim x)$$
$$\forall x, y, z\,(x \sim y \wedge y \sim z \rightarrow x \sim z)$$

and for every $f/n \in \Omega$ the formula

$$\forall x_1, \ldots, x_n, y_1, \ldots, y_n\,(x_1 \sim y_1 \wedge \cdots \wedge x_n \sim y_n \rightarrow f(x_1, \ldots, x_n) \sim f(y_1, \ldots, y_n))$$

and for every $p/n \in \Pi$ the formula

$$\forall x_1, \ldots, x_n, y_1, \ldots, y_n\,(x_1 \sim y_1 \wedge \cdots \wedge x_n \sim y_n \wedge p(x_1, \ldots, x_n) \rightarrow p(y_1, \ldots, y_n)).$$

Let $\tilde{F}$ be the formula that one obtains from F if every occurrence of the equality symbol $\approx$ is replaced by the relation symbol $\sim$.

(a) Let $\mathcal{A}$ be a model of $\tilde{F} \cup Eq(\Sigma)$. Show that the interpretation $\sim_{\mathcal{A}}$ of $\sim$ in $\mathcal{A}$ is a congruence relation. (It is enough if you prove *one* of the properties of congruence relations, say symmetry; the other properties are proved analogously.)

(b) Let $\mathcal{A}$ be a model of $\tilde{F} \cup Eq(\Sigma)$. Use the congruence relation $\sim_{\mathcal{A}}$ to construct a model of F and prove that it is a model.

(c) Prove that a formula F is satisfiable if and only if $Eq(\Sigma) \cup \{\tilde{F}\}$ is satisfiable.

Please submit your solution until Friday, November 18, 2011 at 17:00 by e-mail to `sofronie@uni-koblenz.de` with the keyword “Homework DP” in the subject.

Joint solutions prepared by up to two persons are allowed.
Please do not forget to write your name on your solution!