Universität Koblenz-Landau

FB 4 Informatik

Prof. Dr. Viorica Sofronie-Stokkermans **November 22, 2011**

Exercises for "Decision Procedures for Verification"
Exercise sheet 6

Exercise 6.1: (*2 P*)
Compute a most general unifier of

$$\{\, f(x, g(x)) = y,\ h(y) = h(v),\ v = f(g(z), w) \,\}$$

using the method presented in the lecture (cf. slides from 21.11.2011, page 46; see also the examples on page 47).

Exercise 6.2: (*4 P*)
Let $\Sigma = (\Omega, \Pi)$ with $\Omega = \{b/0, f/1\}$ and $\Pi = \{p/1\}$.

(a) Which is the universe of the Herbrand interpretations over this signature?
If $\mathcal{A}$ is a Herbrand interpretation over Σ how are $b_{\mathcal{A}}$ and $f_{\mathcal{A}}$ defined?

(b) How many different Herbrand interpretations over Σ do exist? Explain briefly.

(c) How many different Herbrand models over Σ does the formula:

$$p(f(f(b))) \wedge \forall x(p(x) \rightarrow p(f(x))) \qquad (1)$$

have? Explain briefly.

(d) Every Herbrand model over Σ of (1) is also a model of

$$\forall x p(f(f(x))) \qquad (2)$$

Give an example of an algebra that is a model of (1) but not of (2).

Exercise 6.3: (*2 P*)
Let $\Sigma = (\Omega, \Pi)$, where $\Omega = \{a/0, f/1, g/1\}$ and $\Pi = \{p/2\}$.

Use the resolution calculus Res (described on page 43 on the slides from 21.11.2011) to show that the following set of clauses (where x, y, z are variables) is unsatisfiable:

$$p(a, z)$$
$$\neg p(f(f(a)), a)$$
$$\neg p(x, g(y)) \vee p(f(x), y)$$

For computing the most general unifiers use the method presented in the lecture (page 46 of the slides from 21.11.2011)

Exercise 6.4: *(2 P)*
Let F be a closed first-order formula with equality over a signature $\Sigma = (\Omega, \Pi)$. Let $\sim \notin \Omega$ be a new binary relation symbol (written as an infix operator). Let the set $Eq(\Sigma)$ contain the formulas

$$\forall x\,(x \sim x)$$
$$\forall x, y\,(x \sim y \rightarrow y \sim x)$$
$$\forall x, y, z\,(x \sim y \land y \sim z \rightarrow x \sim z)$$

and for every $f/n \in \Omega$ the formula

$$\forall x_1, \dots, x_n, y_1, \dots, y_n\,(x_1 \sim y_1 \land \cdots \land x_n \sim y_n \rightarrow f(x_1, \dots, x_n) \sim f(y_1, \dots, y_n))$$

and for every $p/n \in \Pi$ the formula

$$\forall x_1, \dots, x_n, y_1, \dots, y_n\,(x_1 \sim y_1 \land \cdots \land x_n \sim y_n \land p(x_1, \dots, x_n) \rightarrow p(y_1, \dots, y_n)).$$

Let $\tilde{F}$ be the formula that one obtains from F if every occurrence of the equality symbol $\approx$ is replaced by the relation symbol $\sim$.

(a) *Definition.* A binary relation $\sim$ on the support of a Σ-algebra satisfying all properties in $Eq(\Sigma)$ is called a congruence relation.

Let $\mathcal{A}$ be a model of $\tilde{F} \cup Eq(\Sigma)$. Show that the interpretation $\sim_\mathcal{A}$ of $\sim$ in $\mathcal{A}$ is a congruence relation. (It is enough if you prove *one* of the properties of congruence relations, say symmetry; the other properties are proved analogously.)

(b) Let $\mathcal{A}$ be a model of $\tilde{F} \cup Eq(\Sigma)$. Use the congruence relation $\sim_\mathcal{A}$ to construct a model of F and prove that it is a model of F.

Hint: Construct the quotient $\hat{\mathcal{A}} = \mathcal{A}/\sim_\mathcal{A}$ of $\mathcal{A}$ as follows:

- $U_{\hat{\mathcal{A}}} = \{[x] \mid x \in U_\mathcal{A}\}$, where $[x] = \{y \in U_\mathcal{A} \mid y \sim_\mathcal{A} x\}$ is the equivalence class of x w.r.t. $\sim_\mathcal{A}$.
- for every $f/n \in \Omega$, define $f_{\hat{\mathcal{A}}} : U^n_{\hat{\mathcal{A}}} \rightarrow U_{\hat{\mathcal{A}}}$ by $f_{\hat{\mathcal{A}}}([x_1], \dots, [x_n]) = [f_\mathcal{A}(x_1, \dots, x_n)]$.
- for every $p/m \in \Pi$, define $p_{\hat{\mathcal{A}}} \subseteq U^m_{\hat{\mathcal{A}}}$ by: $([x_1], \dots, [x_n]) \in p_{\hat{\mathcal{A}}}$ iff $(x_1, \dots, x_n) \in p_\mathcal{A}$.

(c) Prove that a formula F is satisfiable if and only if $Eq(\Sigma) \cup \{\tilde{F}\}$ is satisfiable.

Please submit your solution until Friday, November 25, 2011 at 17:00 by e-mail to `sofronie@uni-koblenz.de` with the keyword "Homework DP" in the subject.

Joint solutions prepared by up to two persons are allowed.
Please do not forget to write your name on your solution!