Loos-Weispfenning Method:

$$F(x_1, x_2, x_3) = x_1 - x_2 \leq 0 \quad \wedge \quad x_1 - x_3 \leq 0 \quad \wedge \quad -x_1 + x_2 + 2x_3 \leq 0 \wedge -x_3 \leq -1.$$

Task: Is $\exists x_3 \exists x_2 \exists x_1 \, F(x_1, x_2, x_3)$ valid / satisfiable?

Solution: Eliminate quantifiers.

- $\exists x_1 \, F(x_1, x_2, x_3) \equiv \bigvee_{t \in T_{x_1}} F(x_1, x_2, x_3)[t/x]$

  when $F: \begin{cases} x_1 \leq x_2 \quad \wedge \\ x_1 \leq x_3 \quad \wedge \\ x_1 \geq x_2 + 2x_3 \quad \wedge \\ -x_3 \leq -1 \end{cases}$

$\left[\text{where } T_{x_1} = \{-\infty\} \cup \{x_2 + 2x_3\} \right.$

  $\quad\quad\quad\quad\quad \uparrow \quad\quad\quad\quad\quad \uparrow$

  $\quad\quad\quad\quad \text{always} \quad\quad \text{because } x_2 + 2x_3 \text{ is a lower bound for } x_1$

  $\left. \vphantom{\frac{1}{1}} \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \text{in } F. \right]$

Virtual substitution

$\Rightarrow \exists x_1 \, F(x_1, x_2, x_3) \equiv (\underbrace{-\infty \leq x_2}_{T} \wedge \underbrace{-\infty \leq x_3}_{T} \wedge \underbrace{x_2 + 2x_3 \leq -\infty}_{\perp} \wedge -x_3 \leq -1) \vee$

$\quad\quad (x_2 + 2x_3 \leq x_2 \wedge x_2 + 2x_3 \leq x_3 \wedge \underbrace{x_2 + 2x_3 \leq x_2 + 2x_3}_{T} \wedge -x_3 \leq -1)$

$\equiv 2x_3 \leq 0 \wedge x_2 \leq -x_3 \wedge -x_3 \leq -1 \ldots$

- $\exists x_2 \exists x_1 \, F(x_1, x_2, x_3) \equiv \exists x_2 \left(\underbrace{2x_3 \leq 0 \wedge x_2 \leq -x_3 \wedge -x_3 \leq -1}_{F_1} \right)$

$\equiv \bigvee_{t \in T_{x_2}} F_1(x_2, x_3)[t/x_2]$

$\left[\text{where } T_{x_2} = \{-\infty\} \quad\quad (\text{no lower bounds for } x_2) \right]$

Virtual substitution

$\Rightarrow \exists x_2 \exists x_1 \, F(x_1, x_2, x_3) \equiv 2x_3 \leq 0 \wedge \underbrace{-\infty \leq -x_3}_{T} \wedge -x_3 \leq -1$

$\equiv 2x_3 \leq 0 \wedge -x_3 \leq -1.$

- $\exists x_3 \exists x_2 \exists x_1 \, F(x_1, x_2, x_3) \equiv \exists x_3 (2x_3 \leq 0 \wedge -x_3 \leq -1)$

$\equiv \exists x_3 \left(\underbrace{x_3 \leq 0 \wedge x_3 \geq 1}_{F_2} \right)$

$\equiv \bigvee_{t \in T_{x_3}} F_2(x_3)[t/x_3]$

$\left[\text{where } T_{x_3} = \{-\infty\} \cup \{1\} \right.$

$\left. \quad\quad\quad\quad \nwarrow \text{lower bound for } x_3. \right]$

$\Rightarrow \exists x_3 \exists x_2 \exists x_1 \, F(x_1, x_2, x_3) \equiv (\underbrace{-\infty \leq 0}_{T} \wedge \underbrace{-\infty \geq 1}_{\perp}) \vee (\underbrace{1 \leq 0}_{\perp} \wedge \underbrace{1 \geq 1}_{\perp}) \equiv \perp.$