Loos-Weispfenning Method :

$F(x_1,x_2,x_3) = x_1 - x_2 \le 0 \;\wedge\; x_1 - x_3 \le 0 \;\wedge\; -x_1 + x_2 + 2x_3 < 0 \wedge -x_3 \le -1$.

Task : Is $\exists x_3 \exists x_2 \exists x_1 \, F(x_1,x_2,x_3)$ valid / satisfiable ?

Solution : Eliminate quantifiers.

- $\exists x_1 \, F(x_1,x_2,x_3) \equiv \bigvee_{t \in T_{x_1}} F(x_1,x_2,x_3)[t/x]$ when $F: \begin{cases} x_1 \le x_2 \;\wedge \\ x_1 \le x_3 \;\wedge \\ x_1 > x_2 + 2x_3 \;\wedge \\ -x_3 \le -1 \end{cases}$

[where $T_{x_1} = \underset{\text{always}}{\{-\infty\}} \cup \underset{\text{because F contains } x_1 > x_2+2x_3}{\{x_2 + 2x_3 + \varepsilon\}}$]

$\Rightarrow \exists x_1 \, F(x_1,x_2,x_3) \overset{\text{virtual substitution}}{\equiv} (\overbrace{-\infty \le x_2}^{T} \wedge \overbrace{-\infty \le x_3}^{T} \wedge \overbrace{x_2 + 2x_3 < -\infty}^{\perp} \wedge -x_3 \le -1) \vee$
$(x_2 + 2x_3 < x_2 \wedge x_2 + 2x_3 < x_3 \wedge \underbrace{x_2 + 2x_3 \le x_2 + 2x_3}_{T} \wedge -x_3 \le -1)$

(cf slides from 19.12 p.22)

$\equiv 2x_3 < 0 \wedge x_2 < -x_3 \wedge -x_3 \le -1$

- $\exists x_2 \exists x_1 \, F(x_1,x_2,x_3) \equiv \exists x_2 \, (\underbrace{2x_3 < 0 \wedge x_2 < -x_3 \wedge -x_3 \le -1}_{F_1})$

$\equiv \bigvee_{t \in T_{x_2}} F_1(x_2,x_3)[t/x_2]$

[where $T_{x_2} = \{-\infty\}$ (no lower bounds for x_2)]

$\Rightarrow \exists x_2 \exists x_1 \, F(x_1,x_2,x_3) \overset{\text{virtual substitution}}{\equiv} 2x_3 < 0 \wedge \underbrace{-\infty < -x_3}_{T} \wedge -x_3 \le -1$

$\equiv 2x_3 < 0 \wedge -x_3 \le -1$.

- $\exists x_3 \exists x_2 \exists x_1 \, F(x_1,x_2,x_3) \equiv \exists x_3 \, (2x_3 < 0 \wedge -x_3 \le -1)$

$\equiv \exists x_3 \, (\underbrace{x_3 < 0 \wedge x_3 \ge 1}_{F_2})$

$\equiv \bigvee_{t \in T_{x_3}} F_2(x_3)[t/x_3]$

[where $T_{x_3} = \{-\infty\} \cup \{1\}$ ← lower bound for x_3 .]

$\Rightarrow \exists x_3 \exists x_2 \exists x_1 \, F(x_1,x_2,x_3) \equiv (\underbrace{-\infty < 0}_{T} \wedge \underbrace{-\infty \ge 1}_{\perp}) \vee (\underbrace{1 < 0}_{\perp} \wedge 1 \ge 1) \equiv \perp$