Decision Procedures in Verification

Combinations of theories; Combinations of decision procedures

(1)

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Until now:

Logical Theories: generalities

- Theory of Uninterpreted Function Symbols
- Decision procedures for numeric domains

  Difference logic

  Linear arithmetic: Fourier-Motzkin
- Combinations of decision procedures

3.5. Combinations of theories

Problems

The combined decidability problem

For $i = 1, 2$

- let $\mathcal{T}_i$ be a first-order theory in signature Σ_i
- let $\mathcal{L}_i$ be a class of (closed) Σ_i-formulae
- P_i decision procedure for $\mathcal{T}_i$-validity for $\mathcal{L}_i$

Let $\mathcal{T}_1 \bigoplus \mathcal{T}_2$ be a combination of $\mathcal{T}_1$ and $\mathcal{T}_2$
Let $\mathcal{L}_1 \bigoplus \mathcal{L}_2$ be a combination of $\mathcal{L}_1$ and $\mathcal{L}_2$

Question: Can we combine P_1 and P_2 modularly into a decision procedure for the $\mathcal{T}_1 \bigoplus \mathcal{T}_2$-validity problem for $\mathcal{L}_1 \bigoplus \mathcal{L}_2$?

Main issue: How are $\mathcal{T}_1 \bigoplus \mathcal{T}_2$ and $\mathcal{L}_1 \bigoplus \mathcal{L}_2$ defined?

Combinations of theories and models

Forgetting symbols

Let $\Sigma = (\Omega, \Pi)$ and $\Sigma' = (\Omega', \Pi')$ s.t. $\Sigma \subseteq \Sigma'$, i.e., $\Omega \subseteq \Omega'$ and $\Pi \subseteq \Pi'$

For $\mathcal{A} \in \Sigma'$-alg, we denote by $\mathcal{A}_{|\Sigma}$ the Σ-structure for which:

$$U_{\mathcal{A}_{|\Sigma}} = U_{\mathcal{A}}, \quad f_{\mathcal{A}_{|\Sigma}} = f_{\mathcal{A}} \quad \text{for } f \in \Omega;$$
$$P_{\mathcal{A}_{|\Sigma}} = P_{\mathcal{A}} \quad \text{for } P \in \Pi$$

(ignore functions and predicates associated with symbols in $\Sigma' \backslash \Sigma$)

$\mathcal{A}_{|\Sigma}$ is called the restriction (or the reduct) of $\mathcal{A}$ to Σ.

Example: $\Sigma' = (\{+/2, */2, 1/0\}, \{\leq /2, \text{even}/1, \text{odd}/1\})$

$\Sigma = (\{+/2, 1/0\}, \{\leq /2\}) \subseteq \Sigma'$

$\mathcal{N} = (\mathbb{N}, +, *, 1, \leq, \text{even}, \text{odd})$ $\qquad$ $\mathcal{N}_{|\Sigma} = (\mathbb{N}, +, 1, \leq)$

One possibility of combining theories

Syntactic view: $\mathcal{T}_1 + \mathcal{T}_2 = \mathcal{T}_1 \cup \mathcal{T}_2 \subseteq F_{\Sigma_1 \cup \Sigma_2}(X)$

$\mathsf{Mod}(\mathcal{T}_1 \cup \mathcal{T}_2) = \{\mathcal{A} \in (\Sigma_1 \cup \Sigma_2)\text{-alg} \mid \mathcal{A} \models G, \text{ for all } G \text{ in } \mathcal{T}_1 \cup \mathcal{T}_2\}$

where $\Sigma_1 \cup \Sigma_2 = (\Omega_1, \Pi_1) \cup (\Omega_2, \Pi_2) = (\Omega_1 \cup \Omega_2, \Pi_1 \cup \Pi_2)$

One possibility of combining theories

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Semantic view: Let $\mathcal{M}_i = \mathsf{Mod}(\mathcal{T}_i), i = 1, 2$

$\mathcal{M}_1 + \mathcal{M}_2 = \{\mathcal{A} \in (\Sigma_1 \cup \Sigma_2)\text{-alg} \mid \mathcal{A}_{|\Sigma_i} \in \mathcal{M}_i \text{ for } i = 1, 2\}$

One possibility of combining theories

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$$
\begin{array}{lll}
\mathcal{A} \in \mathsf{Mod}(\mathcal{T}_1 \cup \mathcal{T}_2) & \text{iff} & \mathcal{A} \models G, \text{ for all } G \text{ in } \mathcal{T}_1 \cup \mathcal{T}_2 \\
& \text{iff} & \mathcal{A}_{|\Sigma_i} \models G, \text{ for all } G \text{ in } \mathcal{T}_i, i = 1, 2 \\
& \text{iff} & \mathcal{A}_{|\Sigma_i} \in \mathcal{M}_i, i = 1, 2 \\
& \text{iff} & \mathcal{A} \in \mathcal{M}_1 + \mathcal{M}_2
\end{array}
$$

One possibility of combining theories

Syntactic view: $\mathcal{T}_1 + \mathcal{T}_2 = \mathcal{T}_1 \cup \mathcal{T}_2 \subseteq F_{\Sigma_1 \cup \Sigma_2}(X)$

$\mathsf{Mod}(\mathcal{T}_1 \cup \mathcal{T}_2) = \{\mathcal{A} \in (\Sigma_1 \cup \Sigma_2)\text{-alg} \mid \mathcal{A} \models G, \text{ for all } G \text{ in } \mathcal{T}_1 \cup \mathcal{T}_2\}$

Semantic view: Let $\mathcal{M}_i = \mathsf{Mod}(\mathcal{T}_i)$, $i = 1, 2$

$\mathcal{M}_1 + \mathcal{M}_2 = \{\mathcal{A} \in (\Sigma_1 \cup \Sigma_2)\text{-alg} \mid \mathcal{A}_{|\Sigma_i} \in \mathcal{M}_i \text{ for } i = 1, 2\}$

Remark: $\mathcal{A} \in \mathsf{Mod}(\mathcal{T}_1 \cup \mathcal{T}_2)$ iff $(\mathcal{A}_{|\Sigma_1} \in \mathsf{Mod}(\mathcal{T}_1)$ and $\mathcal{A}_{|\Sigma_2} \in \mathsf{Mod}(\mathcal{T}_2))$

Consequence: $\mathsf{Th}(\mathsf{Mod}(\mathcal{T}_1 \cup \mathcal{T}_2)) = \mathsf{Th}(\mathcal{M}_1 + \mathcal{M}_2)$

Example

1. Presburger arithmetic + UIF

$\mathrm{Th}(\mathbb{Z}_+) \cup \mathit{UIF}$ $\qquad$ $\Sigma = (\Omega, \Pi)$

Models: $(A, 0, s, +, \{f_A\}_{f\in\Omega}, \leq, \{P_A\}_{P\in\Pi})$

where $(A, 0, s, +, \leq) \in \mathrm{Mod}(\mathrm{Th}(\mathbb{Z}_+))$.

2. The theory of reals + the theory of a monotone function f

$\mathrm{Th}(\mathbb{R}) \cup \mathrm{Mon}(f)$ $\qquad$ $\mathrm{Mon}(f) : \forall x, y(x \leq y \rightarrow f(x) \leq f(y))$

Models: $(A, +, *, f_A, \{\leq\})$, where

where $(A, +, *, \leq) \in \mathrm{Mod}(\mathrm{Th}(\mathbb{R}))$.

$(A, f_A, \leq) \models \mathrm{Mon}(f)$, i.e. $f_A : A \rightarrow A$ monotone.

Note: The signatures of the two theories share the $\leq$ predicate symbol

Combinations of theories

Definition. A theory is consistent if it has at least one model.

Question: Is the union of two consistent theories always consistent?

Answer: No. (Not even when the two theories have disjoint signatures)

Example: $\Sigma_1 = (\Omega_1, \emptyset), \quad \Sigma_2 = (\{c/0, d/0\}, \emptyset), \quad c, d \notin \Omega_1$

$$\mathcal{T}_1 = \{\exists x, y, z(x \not\approx y \ \wedge \ x \not\approx z \ \wedge \ y \not\approx z)\}$$

$$\mathcal{T}_2 = \{\forall x(x \approx c \ \vee \ x \approx d)\}$$

$$\mathcal{A} \in \mathsf{Mod}(\mathcal{T}_1) \quad \text{iff} \quad |U_{\mathcal{A}}| \geq 3.$$

$$\mathcal{B} \in \mathsf{Mod}(\mathcal{T}_2) \quad \text{iff} \quad |U_{\mathcal{B}}| \leq 2.$$

Combinations of theories

The combined decidability problem

For $i = 1, 2$

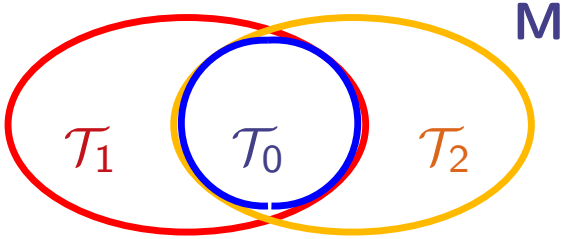
- let $\mathcal{T}_i$ be a first-order theory in signature Σ_i
- assume the $\mathcal{T}_i$ ground satisfiability problem is decidable

Let $\mathcal{T}_1 \bigoplus \mathcal{T}_2$ be a combination of $\mathcal{T}_1$ and $\mathcal{T}_2$

Question:

Is the $\mathcal{T}_1 \bigoplus \mathcal{T}_2$ ground satisfiability problem decidable?

Goal: Modularity



Modular Reasoning

$\mathcal{T}_0$: Σ_0-theory.

$\mathcal{T}_i$: Σ_i-theory; $\quad \mathcal{T}_0 \subseteq \mathcal{T}_i \quad \Sigma_0 \subseteq \Sigma_i$.

Example:

lists$(\mathbb{R}) \cup$ **arrays**$(\mathbb{R})$

Can use provers for $\mathcal{T}_1, \mathcal{T}_2$ as blackboxes to prove theorems in $\mathcal{T}_1 \cup \mathcal{T}_2$?

Which information needs to be exchanged between the provers?

Combinations of theories

For $i = 1, 2$

- let $\mathcal{T}_i$ be a first-order theory in signature Σ_i
- s.t. the ground satisfiability problem for $\mathcal{T}_i$ is decidable

Question: Is the ground satisfiability problem for $\mathcal{T}_1 \cup \mathcal{T}_2$ decidable?

Combinations of theories

For $i = 1, 2$
- let $\mathcal{T}_i$ be a first-order theory in signature Σ_i
- s.t. the ground satisfiability problem for $\mathcal{T}_i$ is decidable

Question: Is the ground satisfiability problem for $\mathcal{T}_1 \cup \mathcal{T}_2$ decidable?

In general: No (restrictions needed for affirmative answer)

Example. Word problem for $\mathcal{T}$: Decide if $\mathcal{T} \models \forall x(s \approx t)$

$\mathcal{A}$: theory of associativity

$\mathcal{G}$ finite set of ground equations
(presentation for semigroup
with undecidable word problem)

$\uparrow$

($\exists$ finitely-presented semigroup with
undecidable word problem [Matijasevic'67])

Word problem: decidable for $\mathcal{A}, \mathcal{G}$; undecidable for $\mathcal{A} \cup \mathcal{G}$

Combinations of theories

For $i = 1, 2$
- let $\mathcal{T}_i$ be a first-order theory in signature Σ_i
- s.t. the ground satisfiability problem for $\mathcal{T}_i$ is decidable

Question: Is the ground satisfiability problem for $\mathcal{T}_1 \cup \mathcal{T}_2$ decidable?

In general: No (restrictions needed for affirmative answer)

Example. Word problem for $\mathcal{T}$: Decide if $\mathcal{T} \models \forall x(s \approx t)$

Simpler instances: combinations of theories over disjoint signatures, theories sharing constructors, compatibility with shared theory ...

Combinations of theories

For $i = 1, 2$
- let $\mathcal{T}_i$ be a first-order theory in signature Σ_i
- s.t. the ground satisfiability problem for $\mathcal{T}_i$ is decidable

Question: Is the ground satisfiability problem for $\mathcal{T}_1 \cup \mathcal{T}_2$ decidable?

In general: No (restrictions needed for affirmative answer)

> **Theorem** [Bonacina, Ghilardi et.al, IJCAR 2006]
> There are theories $\mathcal{T}_1, \mathcal{T}_2$ with disjoint signatures and decidable ground satisfiability problem such that ground satisfiability in $\mathcal{T}_1 \cup \mathcal{T}_2$ is undecidable.

Idea: Construct $\mathcal{T}_1$ such that ground satisfiability is decidable, but it is undecidable whether a constraint Γ_1 is satisfiable in an infinite model of $\mathcal{T}_1$. (Construction uses Turing Machines). Let $\mathcal{T}_2$ having only infinite models.

Combination of theories over disjoint signatures

The Nelson/Oppen procedure

Given: $\mathcal{T}_1, \mathcal{T}_2$ first-order theories with signatures Σ_1, Σ_2

Assume that $\Sigma_1 \cap \Sigma_2 = \emptyset$ (share only $\approx$)

P_i decision procedures for satisfiability of ground formulae w.r.t. $\mathcal{T}_i$

ϕ quantifier-free formula over $\Sigma_1 \cup \Sigma_2$

Task: Check whether ϕ is satisfiable w.r.t. $\mathcal{T}_1 \cup \mathcal{T}_2$

Note: Restrict to conjunctive quantifier-free formulae

$\phi \mapsto DNF(\phi)$

$DNF(\phi)$ satisfiable in $\mathcal{T}$ iff one of the disjuncts satisfiable in $\mathcal{T}$

Example

[Nelson & Oppen, 1979]

Theories

$\mathcal{R}$	theory of rationals	$\Sigma_{\mathcal{R}} = \{\leq, +, -, 0, 1\}$	$\approx$
$\mathcal{L}$	theory of lists	$\Sigma_{\mathcal{L}} = \{\mathsf{car}, \mathsf{cdr}, \mathsf{cons}\}$	$\approx$
$\mathcal{E}$	theory of equality (UIF)	Σ: free function and predicate symbols	$\approx$

Example

[Nelson & Oppen, 1979]

Theories

$\mathcal{R}$	theory of rationals	$\Sigma_{\mathcal{R}} = \{\leq, +, -, 0, 1\}$	$\approx$
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$\mathcal{E}$	theory of equality (UIF)	Σ: free function and predicate symbols	$\approx$

Problems:

1. $\mathcal{R} \cup \mathcal{L} \cup \mathcal{E} \models \forall x, y (x \leq y \wedge y \leq x + \mathsf{car}(\mathsf{cons}(0, x)) \wedge P(h(x) - h(y)) \rightarrow P(0))$
2. Is the following conjunction:

$$c \leq d \ \wedge \ d \leq c + \mathsf{car}(\mathsf{cons}(0, c)) \ \wedge \ P(h(c) - h(d)) \ \wedge \ \neg P(0)$$

satisfiable in $\mathcal{R} \cup \mathcal{L} \cup \mathcal{E}$?

An Example

	$\mathcal{R}$	$\mathcal{L}$	$\mathcal{E}$
Σ	$\{\leq, +, -, 0, 1\}$	$\{\mathsf{car}, \mathsf{cdr}, \mathsf{cons}\}$	$F \cup P$
Axioms	$x + 0 \approx x$	$\mathsf{car}(\mathsf{cons}(x, y)) \approx x$	
	$x - x \approx 0$	$\mathsf{cdr}(\mathsf{cons}(x, y)) \approx y$	
(univ.	$+$ is A, C	$\mathsf{at}(x) \vee \mathsf{cons}(\mathsf{car}(x), \mathsf{cdr}(x)) \approx x$	
quantif.)	$\leq$ is R, T, A	$\neg\mathsf{at}(\mathsf{cons}(x, y))$	
	$x \leq y \vee y \leq x$		
	$x \leq y \rightarrow x + z \leq y + z$		

Is the following conjunction:

$$c \leq d \ \wedge \ d \leq c + \mathsf{car}(\mathsf{cons}(0, c)) \ \wedge \ P(h(c) - h(d)) \ \wedge \ \neg P(0)$$

satisfiable in $\mathcal{R} \cup \mathcal{L} \cup \mathcal{E}$?

Step 1: Purification

$$c \leq d \ \wedge \ d \leq c + \mathsf{car}(\mathsf{cons}(0, c)) \ \wedge \ P(h(c) - h(d)) \ \wedge \ \neg P(0)$$

Step 1: Purification

$$c \leq d \ \wedge \ d \leq c + \underbrace{\mathsf{car}(\mathsf{cons}(0, c))}_{c_1} \ \wedge \ P(h(c) - h(d)) \ \wedge \ \neg P(0)$$

Step 1: Purification

$$c \leq d \ \wedge \ d \leq c + \underbrace{\mathsf{car}(\mathsf{cons}(0, c))}_{c_1} \ \wedge \ P(\underbrace{h(c) - h(d)}_{c_2}) \ \wedge \ \neg P(0)$$

Step 1: Purification

$$c \leq d \;\wedge\; d \leq c + \underbrace{\mathsf{car}(\mathsf{cons}(0, c))}_{c_1} \;\wedge\; P(\underbrace{\underbrace{h(c)}_{c_3} - \underbrace{h(d)}_{c_4}}_{c_2}) \;\wedge\; \neg P(\underbrace{0}_{c_5})$$

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$\mathcal{R}$	$\mathcal{L}$	$\mathcal{E}$
$c \leq d$	$c_1 \approx \mathsf{car}(\mathsf{cons}(c_5, c))$	$P(c_2)$
$d \leq c + c_1$		$\neg P(c_5)$
$c_2 \approx c_3 - c_4$		$c_3 \approx h(c)$
$c_5 \approx 0$		$c_4 \approx h(d)$

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$\mathcal{R}$	$\mathcal{L}$	$\mathcal{E}$
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$d \leq c + c_1$		$\neg P(c_5)$
$c_2 \approx c_3 - c_4$		$c_3 \approx h(c)$
$c_5 \approx 0$		$c_4 \approx h(d)$
satisfiable	satisfiable	satisfiable

Step 2: Propagation

$$c \leq d \ \wedge \ d \leq c + \underbrace{\mathsf{car}(\mathsf{cons}(0, c))}_{c_1} \ \wedge \ P(\underbrace{\underbrace{h(c)}_{c_3} - \underbrace{h(d)}_{c_4}}_{c_2}) \ \wedge \ \neg P(\underbrace{0}_{c_5})$$

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$c_2 \approx c_3 - c_4$		$c_3 \approx h(c)$
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deduce and propagate equalities between constants entailed by components

Step 2: Propagation

$$c \leq d \;\wedge\; d \leq c + \underbrace{\mathsf{car}(\mathsf{cons}(0, c))}_{c_1} \;\wedge\; P(\underbrace{\underbrace{h(c)}_{c_3} - \underbrace{h(d)}_{c_4}}_{c_2}) \;\wedge\; \neg P(\underbrace{0}_{c_5})$$

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$d \leq c + c_1$		$\neg P(c_5)$
$c_2 \approx c_3 - c_4$		$c_3 \approx h(c)$
$c_5 \approx 0$		$c_4 \approx h(d)$

$$c_1 \approx c_5$$

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$c_2 \approx c_3 - c_4$		$c_3 \approx h(c)$
$c_5 \approx 0$		$c_4 \approx h(d)$
$c_1 \approx c_5$	$c_1 \approx c_5$	
$c \approx d$		

Step 2: Propagation

$$c \leq d \;\wedge\; d \leq c + \underbrace{\mathsf{car}(\mathsf{cons}(0, c))}_{c_1} \;\wedge\; P(\underbrace{\underbrace{h(c)}_{c_3} - \underbrace{h(d)}_{c_4}}_{c_2}) \;\wedge\; \neg P(\underbrace{0}_{c_5})$$

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$c_2 \approx c_3 - c_4$		$c_3 \approx h(c)$
$c_5 \approx 0$		$c_4 \approx h(d)$
$c_1 \approx c_5$	$c_1 \approx c_5$	$c \approx d$
$c \approx d$		$c_3 \approx c_4$

Step 2: Propagation

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$c_2 \approx c_3 - c_4$		$c_3 \approx h(c)$
$c_5 \approx 0$		$c_4 \approx h(d)$
$c_1 \approx c_5$	$c_1 \approx c_5$	$c \approx d$
$c \approx d$		$c_3 \approx c_4$
$c_2 \approx c_5$		$\bot$

The Nelson-Oppen algorithm

ϕ conjunction of literals

Step 1. Purification $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \phi \mapsto (\mathcal{T}_1 \cup \phi_1) \cup (\mathcal{T}_2 \cup \phi_2)$:

where ϕ_i is a pure Σ_i-formula and $\phi_1 \wedge \phi_2$ is equisatisfiable with ϕ.

Informally: "Separate" the formula ϕ into two "pure" formulae using renaming.

Step 2. Propagation.

The decision procedure for ground satisfiability for $\mathcal{T}_1$ and $\mathcal{T}_2$ fairly exchange information concerning entailed unsatisfiability of constraints in the shared signature

i.e. clauses over the shared variables.

until an inconsistency is detected or a saturation state is reached.

Informally: Provers for component theories exchange information about shared symbols.

Step 1: Purification

Given: ϕ conjunctive quantifier-free formula over $\Sigma_1 \cup \Sigma_2$

Task: Find ϕ_1, ϕ_2 s.t. ϕ_i is a pure Σ_i-formula and $\phi_1 \wedge \phi_2$ equivalent with ϕ

$$
\begin{array}{lcl}
f(s_1, \ldots, s_n) \approx g(t_1, \ldots, t_m) & \mapsto & u \approx f(s_1, \ldots, s_n) \wedge u \approx g(t_1, \ldots, t_m) \\
f(s_1, \ldots, s_n) \not\approx g(t_1, \ldots, t_m) & \mapsto & u \approx f(s_1, \ldots, s_n) \wedge v \approx g(t_1, \ldots, t_m) \wedge u \not\approx v \\
(\neg) P(\ldots, s_i, \ldots) & \mapsto & (\neg) P(\ldots, u, \ldots) \wedge u \approx s_i \\
(\neg) P(\ldots, s_i[t], \ldots) & \mapsto & (\neg) P(\ldots, s_i[t \mapsto u], \ldots) \wedge u \approx t
\end{array}
$$

where $t \approx f(t_1, \ldots, t_n)$

Termination: Obvious

Correctness: $\phi_1 \wedge \phi_2$ and ϕ equisatisfiable.

The Nelson-Oppen algorithm

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where ϕ_i is a pure Σ_i-formula and $\phi_1 \wedge \phi_2$ is equisatisfiable with ϕ.

not problematic; requires linear time

Step 2. Propagation.

The decision procedure for ground satisfiability for $\mathcal{T}_1$ and $\mathcal{T}_2$ fairly exchange information concerning entailed unsatisfiability of constraints in the shared signature

i.e. clauses over the shared variables.

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The decision procedure for ground satisfiability for $\mathcal{T}_1$ and $\mathcal{T}_2$ fairly exchange information concerning entailed unsatisfiability of constraints in the shared signature

i.e. clauses over the shared variables.

until an inconsistency is detected or a saturation state is reached.

not problematic; termination guaranteed

Sound: if inconsistency detected input unsatisfiable

Complete: under additional assumptions

Implementation

ϕ conjunction of literals

Step 1. Purification: $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \phi \mapsto (\mathcal{T}_1 \cup \phi_1) \cup (\mathcal{T}_2 \cup \phi_2)$,
where ϕ_i is a pure Σ_i-formula and $\phi_1 \wedge \phi_2$ is equisatisfiable with ϕ.

Step 2. Propagation: The decision procedure for ground satisfiability for $\mathcal{T}_1$ and $\mathcal{T}_2$ fairly exchange information concerning entailed unsatisfiability of constraints in the shared signature
i.e. clauses over the shared variables.

until an inconsistency is detected or a saturation state is reached.

How to implement Propagation?

Guessing: guess a maximal set of literals containing the shared variables; check it for $\mathcal{T}_i \cup \phi_i$ consistency.

Backtracking: identify disjunction of equalities between shared variables entailed by $\mathcal{T}_i \cup \phi_i$; make case split by adding some of these equalities to ϕ_1, ϕ_2. Repeat as long as possible.

Implementation of propagation

Guessing variant

Guess a maximal set of literals containing the shared variables V (arrangement: $\alpha(V, E) = (\bigwedge_{(u,v)\in E} u \approx v \wedge \bigwedge_{(u,v)\notin E} u \not\approx v)$, where E equivalence relation); check it for $\mathcal{T}_i \cup \phi_i$ consistency.

On the blackboard: Example 10.5 and 10.7 pages 272, 273
Example 10.6 and 10.9 pages 272, 275

from the book "The Calculus of Computation" by A. Bradley and Z. Manna

Advantage: Whenever constraints are represented as Boolean combinations of atoms, one may combine heuristics of SMT solvers with specific features of the theories to be combined to produce the right arrangement efficiently.

Implementation of propagation

Backtracking variant

Identify disjunction of equalities between shared variables entailed by $\mathcal{T}_i \cup \phi_i$; make case split by adding some of these equalities to ϕ_1, ϕ_2.

Repeat as long as possible.

On the blackboard: Example 10.14, page 280-281, and Example 10.13, page 279, from the book "The Calculus of Computation" by A. Bradley and Z. Manna

Advantages:

- it works on the non-disjoint case as well

- can be made deterministic for combinations of convex theories

$\mathcal{T}$ convex iff whenever $\mathcal{T} \models \bigwedge_{i=1}^{n} A_i \rightarrow \bigvee_{j=1}^{m} B_j$
there exists j s.t. $\mathcal{T} \models \bigwedge_{i=1}^{n} A_i \rightarrow B_j$