

Difference Logic: Examples

Let $S_1 = x - y < 5 \wedge y - z < 3 \wedge z - x \leq -6$

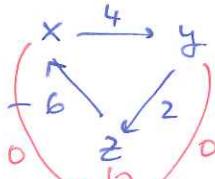
1. Check the satisfiability of S_1 over \mathbb{Z}

Note: When checking satisfiability over \mathbb{Z} we use the fact that $x_1 - x_2 \leq k$ iff $x_1 - x_2 \leq k - 1$

S_1 can be rewritten to the following set of constraints in positive difference logic:

$$x - y \leq 4 \wedge y - z \leq 2 \wedge z - x \leq -6.$$

Associated graph:



no negative cycles.
⇒ formula is satisfiable.

Satisfying assignment:

To compute a satisfying assignment, we add a new vertex s and edges with weight 0 from any node to s .

Let $\delta(x, s)$ be the length of the shortest path from $x \in \{x, y, z\}$ to s .

Then $\beta: X \rightarrow \mathbb{Z}$ defined by $\beta(x) = \delta(x, s)$ for all $x \in X$ is a satisfying assignment

$$\beta(x) = \delta(x, s) = 0; \quad \beta(y) = \delta(y, s) = 4; \quad \beta(z) = \delta(z, s) = -6.$$

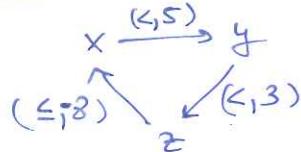
Check if β is solution

$$\beta(x) - \beta(y) = 0 - 4 = 4 < 5; \quad \beta(y) - \beta(z) = 4 - (-6) = 2 < 3; \quad \beta(z) - \beta(x) = -6 \leq -6.$$

let $S_2 = x - y < 5 \wedge y - z < 3 \wedge z - x \leq -8$

2. Check the satisfiability of S_2 over \mathbb{Q}

Associated graph:

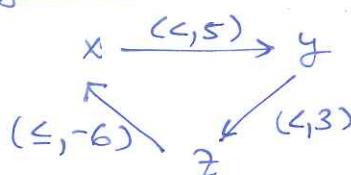


weight of the cycle $x \rightarrow y \rightarrow z \rightarrow x$
is $(<, 5) + (<, 3) + (< -8) = (<, 0)$
⇒ negative cycle.

S_2 is unsatisfiable.

3. Check the satisfiability of S_1 over \mathbb{Q}

Associated graph:



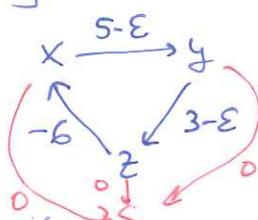
no negative cycles
⇒ formula is satisfiable.

Satisfying assignment:

let ε be sufficiently small. we rewrite S_1 as:

$$\begin{cases} x - y \leq 5 - \varepsilon \\ y - z \leq 3 - \varepsilon \\ z - x \leq -6 \end{cases}$$

Associated graph:



for ε sufficiently small: no negative cycles.

$$\delta(x, s) = 0, \delta(z, s) = -6$$

$$\delta(y, s) = -3 - \varepsilon.$$

Satisfying assignment: $\beta(x) = 0, \beta(y) = -3 - \varepsilon, \beta(z) = -6$ for every small $\varepsilon > 0$.

Check if β solution

$$\beta(x) - \beta(y) = -(-3 - \varepsilon) = 3 + \varepsilon < 5; \quad \beta(y) - \beta(z) = -3 - \varepsilon - (-6) = 3 - \varepsilon < 3; \quad \beta(z) - \beta(x) = -6 \leq -6.$$

$$\text{e.g. } \varepsilon = \frac{1}{100000}$$