

## FOURIER-MOTZKIN METHOD: Example.

Let  $F(x_1, x_2, x_3) = (x_1 - x_2 \leq 0 \wedge x_1 - x_3 \leq 0 \wedge -x_1 + x_2 + 2x_3 \leq 0, -x_3 \leq -1)$

Task: Check whether  $F$  is satisfiable.

Fact:  $F$  is satisfiable  $\Leftrightarrow \underbrace{\exists x_3 \exists x_2 \exists x_1 F(x_1, x_2, x_3)}_{(*)}$  is valid.

Solution: In order to check the validity of  $(*)$  we successively eliminate the quantified variables.

1) Eliminate  $\exists x_1$ :

$$\begin{aligned} \exists x_1 F(x_1, x_2, x_3) &\equiv \exists x_1 (x_1 \leq x_2 \wedge \\ &\quad x_1 \leq x_3 \wedge \\ &\quad x_1 \geq x_2 + 2x_3 \wedge \\ &\quad 0 \leq x_3 - 1) \end{aligned}$$

(all inequalities in  $F$  are written such that  $x_1$  is isolated on the left-hand side of inequalities)

$$\begin{aligned} &\equiv x_2 + 2x_3 \leq x_2 \wedge \\ &\quad x_2 + 2x_3 \leq x_3 \wedge \\ &\quad 0 \leq x_3 - 1. \end{aligned}$$

(combine upper-lower bounds of  $x_1$ ; use transitivity)

2) Eliminate  $\exists x_2$ :

$$\begin{aligned} \exists x_2 \exists x_1 F(x_1, x_2, x_3) &\equiv \exists x_2 (x_2 + 2x_3 \leq x_2 \wedge \\ &\quad x_2 + 2x_3 \leq x_3 \wedge \\ &\quad 0 \leq x_3 - 1) \end{aligned}$$

$$\begin{aligned} &\equiv \exists x_2 (0 \geq 2x_3 \wedge \\ &\quad x_2 \leq -x_3 \wedge \\ &\quad 0 \leq x_3 - 1) \end{aligned}$$

$$\equiv 0 \geq 2x_3 \wedge 0 \leq x_3 - 1. \quad (\text{no lower bound for } x_2.)$$

3) Eliminate  $\exists x_3$ :

$$\exists x_3 \exists x_2 \exists x_1 F(x_1, x_2, x_3) \equiv \exists x_3 (0 \geq 2x_3 \wedge 0 \leq x_3 - 1)$$

$$\equiv \exists x_3 (x_3 \leq 0 \wedge x_3 \geq 1)$$

$$\equiv 1 \leq 0$$

$$\equiv \perp \quad (\text{false}).$$

(combine lower/upper bound & use trans.)

Conclusion  $F$  is unsatisfiable