

Decision Procedures in Verification

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Motivation

Long-term goal of research in computer science

- use computers as 'intelligent assistants' in
 - e.g. mathematics, engineering (and other fields)

Main problem

- complex description of problems to be solved
 - ↦ complex systems, complex encoding

Examples of application domains

MATHEMATICS
Tasks <ul style="list-style-type: none">– construct proofs– check proofs
Theories <ul style="list-style-type: none">– numbers– polynomials– functions over numeric domains– algebras

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Example: Lipschitz functions

$$\mathbb{R} \cup (\mathcal{L}_{c,\lambda_1}^f) \cup (\mathcal{L}_{c,\lambda_2}^g) \models (\mathcal{L}_{c,(\lambda_1+\lambda_2)}^{f+g})$$

$$(\mathcal{L}_{c,\lambda_1}^f) \quad \forall x \ |f(x) - f(c)| \leq \lambda_1 \cdot |x - c|$$

$$(\mathcal{L}_{c,\lambda_2}^g) \quad \forall x \ |g(x) - g(c)| \leq \lambda_2 \cdot |x - c|$$

$$(\mathcal{L}_{c,(\lambda_1+\lambda_2)}^{f+g}) \quad \forall x \ |f(x)+g(x)-f(c)-g(c)| \leq (\lambda_1 + \lambda_2) \cdot |x - c|$$

Similar:

- free functions; (piecewise) monotone functions
- functions defined according to a partition of their domain of definition, ...

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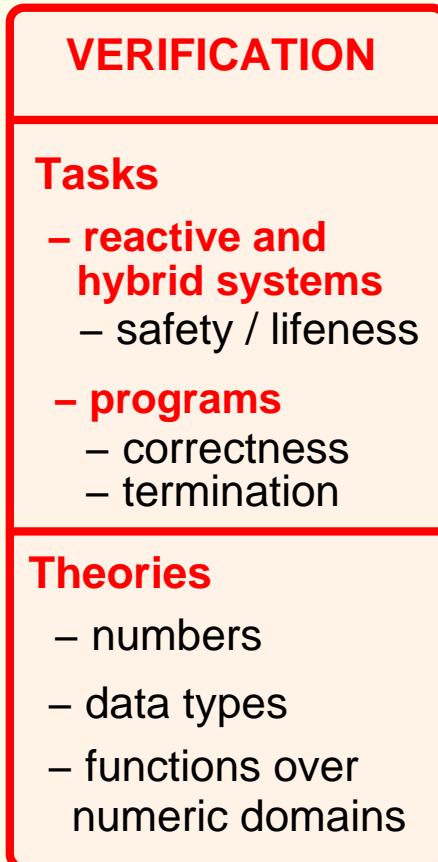
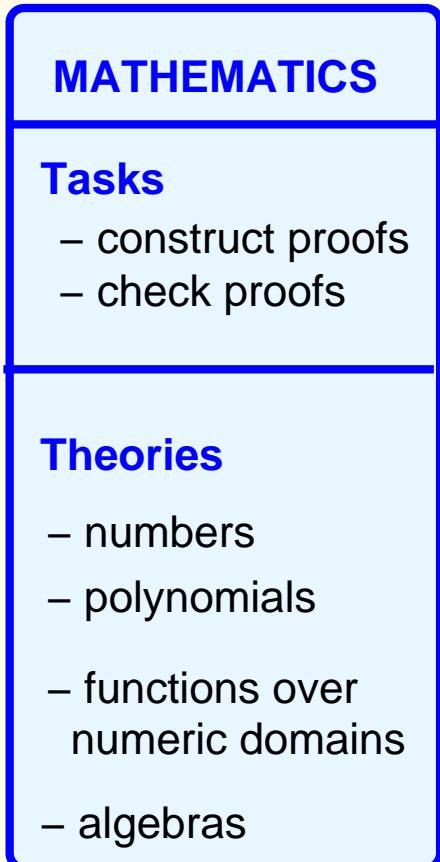
VERIFICATION
Tasks <ul style="list-style-type: none">– reactive and hybrid systems– safety / liveness– programs– correctness– termination

Infinite state systems (software, real time, hybrid)

- simulation/testing cannot guarantee absence of errors
 → need symbolic methods

Solution: - Build 'formal model' of the system;
 - **Prove** that properties are 'consequences of the model'

Examples of application domains



Example: Does BUBBLE SORT return a sorted array?

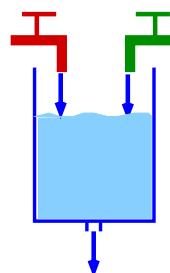
```
int [] BUBBLESORT(int[] a) {  
    int i, j, t;  
    for (i := |a| - 1; i > 0; i := i - 1) {  
        for (j := 0; j < i; j := j + 1) {  
            if (a[j] > a[j + 1]) {t := a[j];  
                a[j] := a[j + 1];  
                a[j + 1] := t};  
        } } return a}
```

Examples of application domains

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VERIFICATION
Tasks <ul style="list-style-type: none">– programs correctness/termination– reactive/hybrid systems safety/liveness– cryptography correctness crypt. prot.
Theories <ul style="list-style-type: none">– numbers– data types– functions over numeric domains

Controllers



Check:

- No overflow
- Substances in the right proportion
- If substances in wrong proportion, tank can be drained in ≤ 200 s.

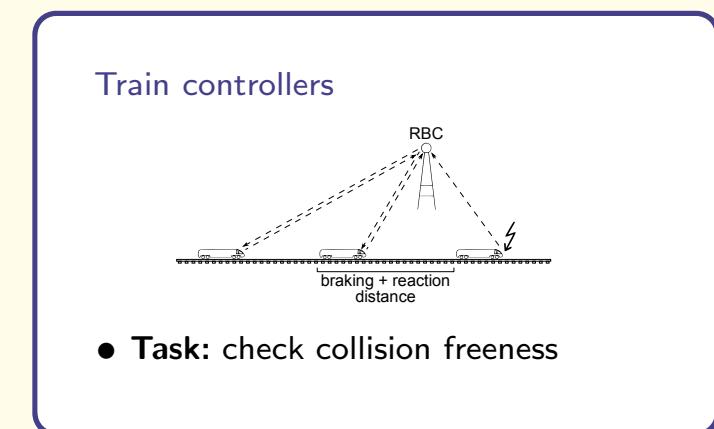
Determine values for parameters
such that this is the case

Examples of application domains

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Controllers



Examples of application domains

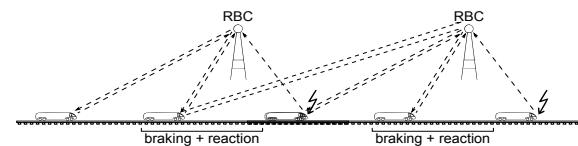
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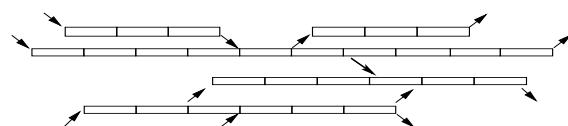
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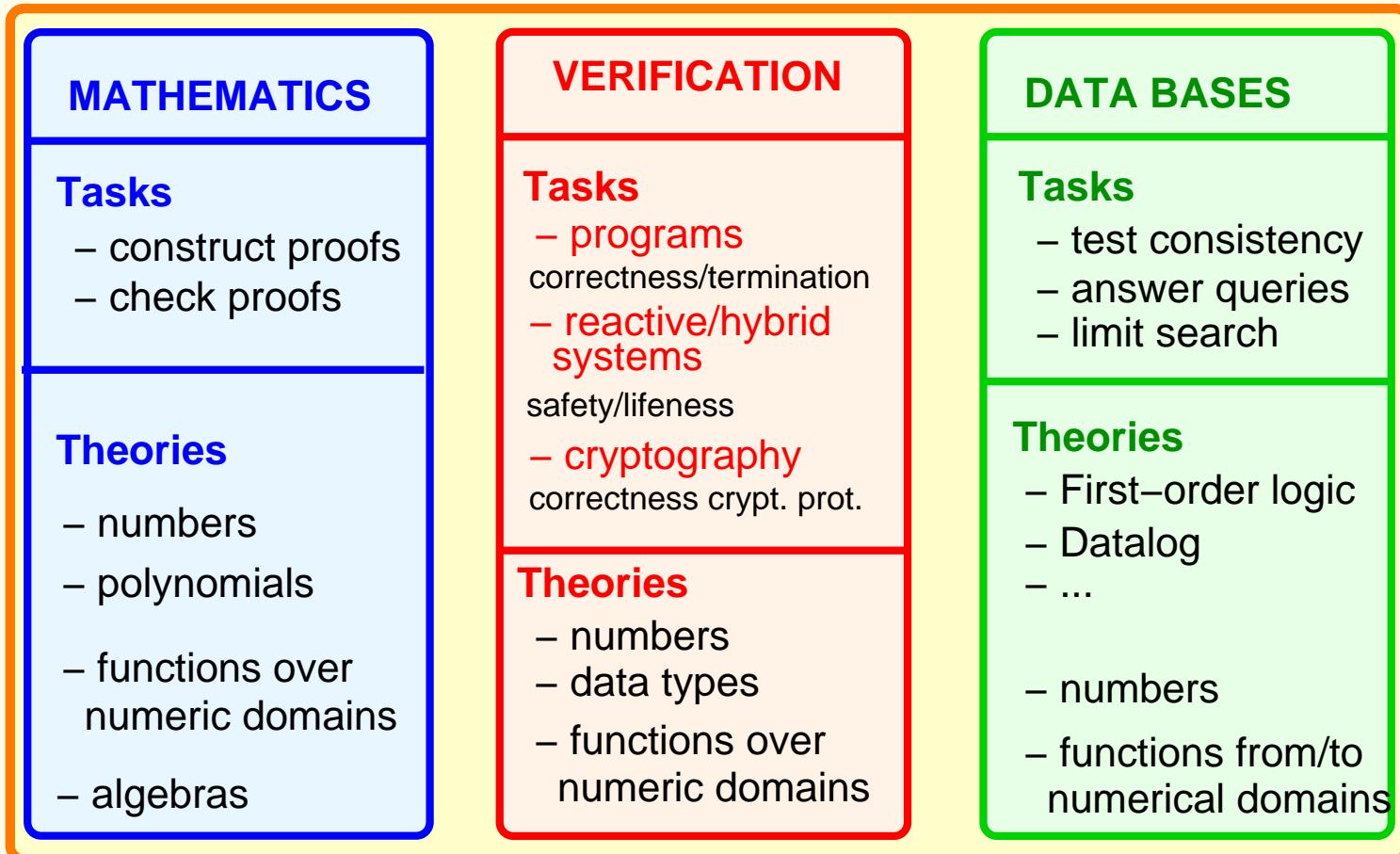
- Two or more controllers
- non-disjoint sets of controlled trains
 - synchronization for the control of common trains



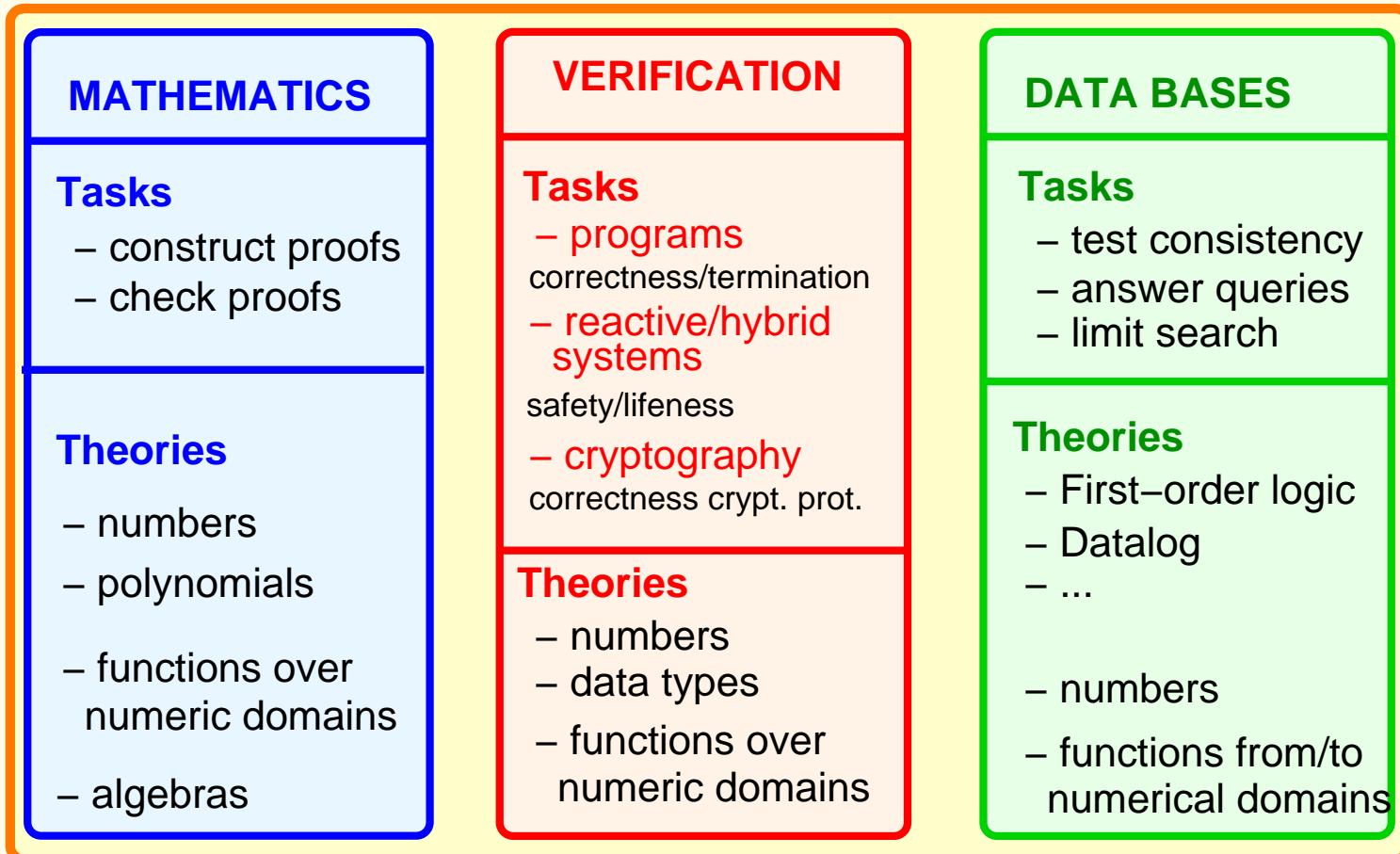
- complex track topology



Examples of application domains

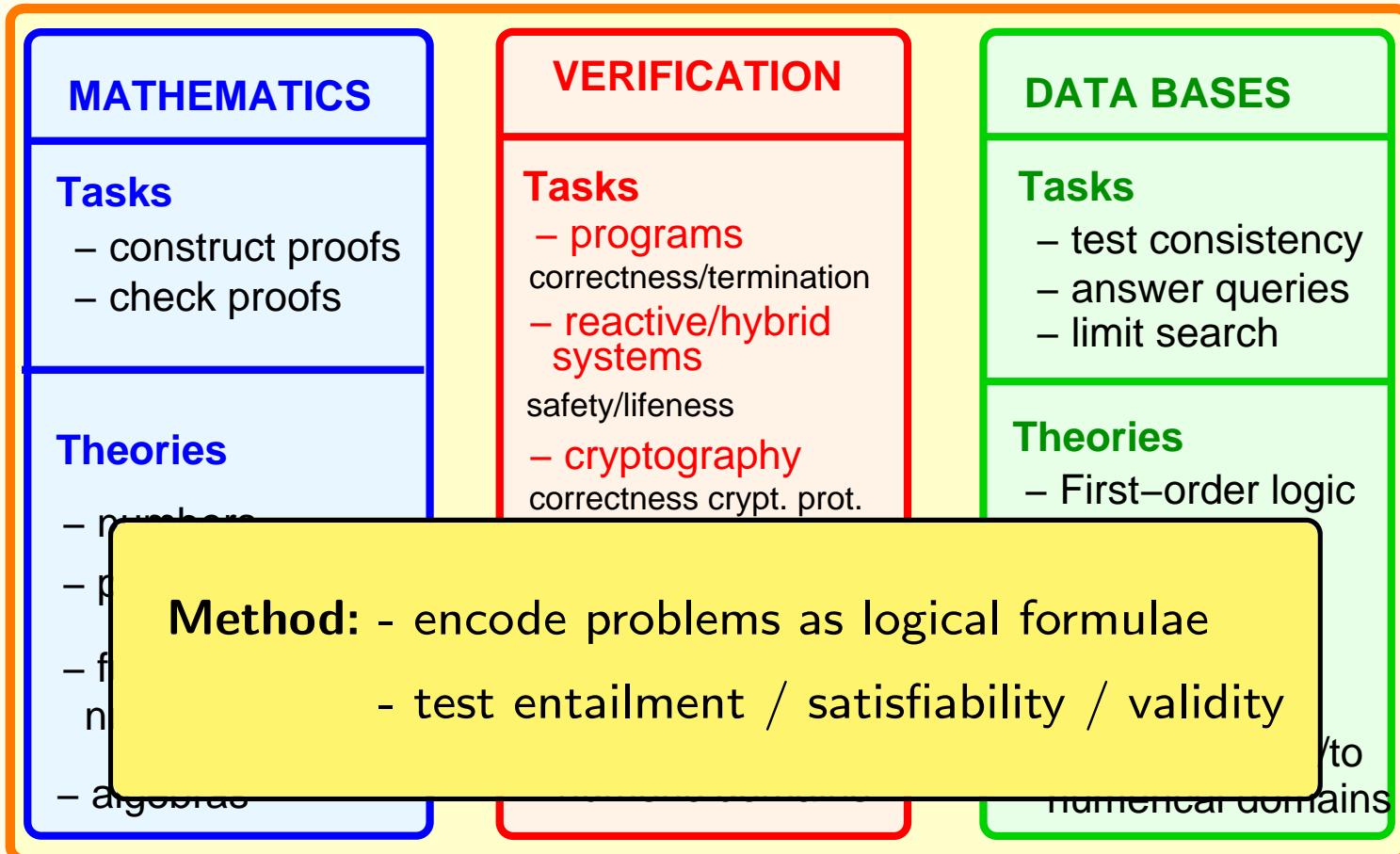


Examples of application domains



complex systems (MAS, reactive systems w. embedded software, databases)

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Problems and goals

- 1st order logic is undecidable: cannot build an 'all-purpose' program
- + often fragments of theories occurring in applications are decidable

- theories do not occur alone: need to consider combinations of theories
- + often provers for the component theories can be combined efficiently

Problems and goals

- 1st order logic is undecidable: cannot build an 'all-purpose' program
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- theories do not occur alone: need to consider combinations of theories
- + often provers for the component theories can be combined efficiently

Important:

Identify theories (and extensions/combinations thereof) which are decidable (with low complexity) and relevant in applications

Overview

Plan of the lecture:

- Reasoning in standard theories
- Reasoning in theory extensions
- Reasoning in combinations of theories

Important: identify decidable/tractable fragments

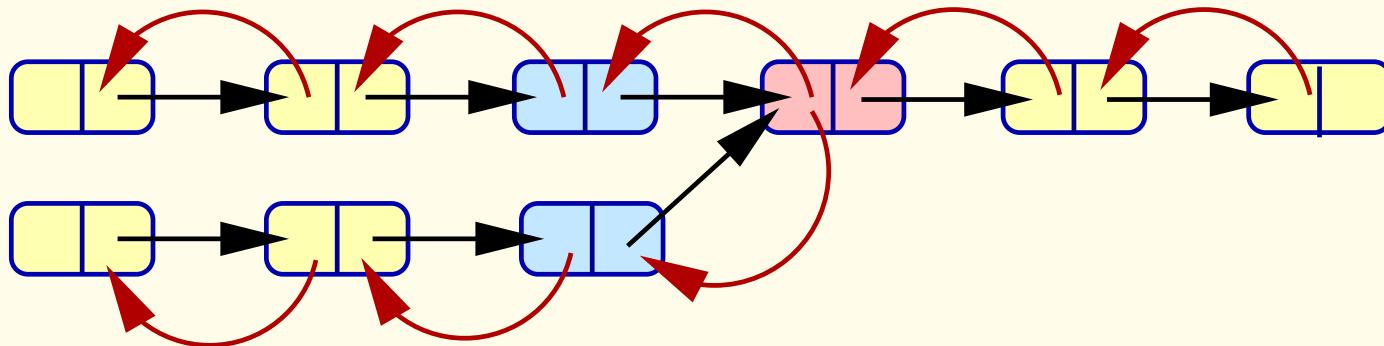
Reasoning about standard datatypes

- **Numbers** - natural numbers, integers, reals, rationals
- **Data structures** - theories of lists
- theory of acyclic lists
- theory of arrays
- theories of sets, multisets
- **Algebraic theories** - (total/partial) orderings
- lattices, semilattices
- distributive lattices
- Boolean algebras
- groups, rings, fields, ...

Reasoning in theory extensions

- **Numbers** - integers, reals, rationals
- **Data structures**
 - theories of lists of integers, reals, ...
 - theory of acyclic lists of integers, reals, ...
 - theory of arrays of integers, reals, ...
 - theories of sets of integers, reals, ...
 - + functions (free, rec. def.) e.g : length, card
- **Algebraic theories**
 - (total/partial) orderings with monotone functions
 - lattices, semilattices with operators
 - distributive lattices with operators
 - Boolean algebras with operators
 - fields with operators

Example: A theory of doubly-linked lists

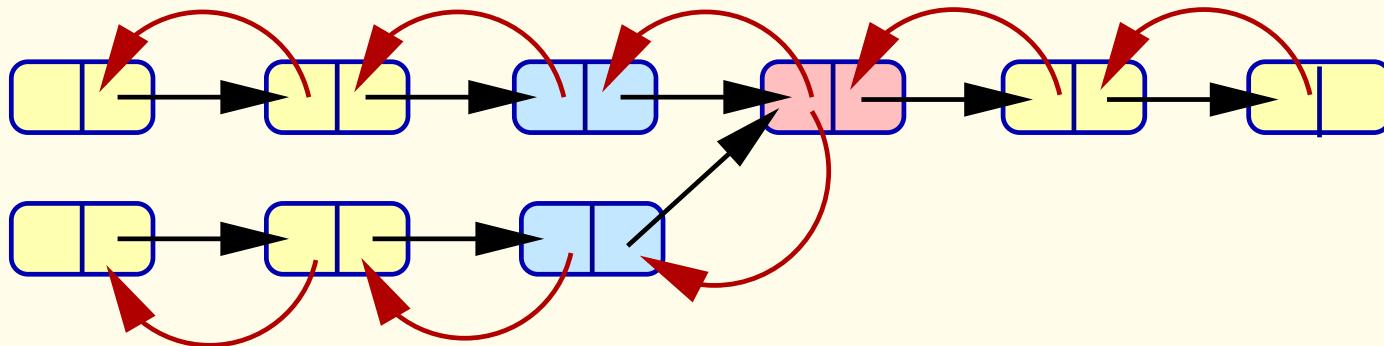


$$\forall p \ (p \neq \text{null} \wedge p.\text{next} \neq \text{null} \rightarrow p.\text{next}.\text{prev} = p)$$

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$$\wedge \ c \neq \text{null} \wedge c.\text{next} \neq \text{null} \wedge d \neq \text{null} \wedge d.\text{next} \neq \text{null} \wedge c.\text{next} = d.\text{next} \wedge c \neq d \models \perp$$

Example: A theory of doubly-linked lists



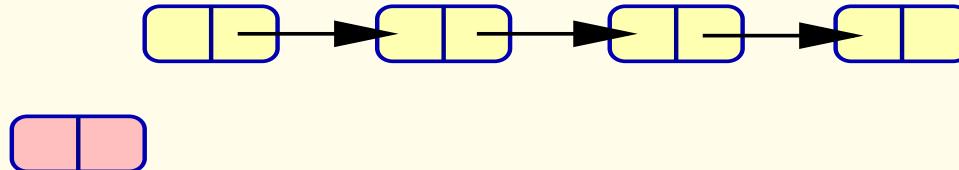
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Scalar fields + additional axioms:

$$\forall p (p \neq \text{null} \wedge p.\text{next} \neq \text{null} \rightarrow p.\text{info} \leq p.\text{next}.\text{info})$$

Example: List insertion



Initially list is sorted: $p.\text{next} \neq \text{null} \rightarrow p.\text{info} \geq p.\text{next.info}$

$c.\text{info} = x, c.\text{next} = \text{null}$

for all $p \neq c$ **do**

if $p.\text{info} \leq x$ **then if** $\text{First}(p)$ **then** $c.\text{next}' = p$, $\text{First}'(c)$, $\neg\text{First}'(p)$ **endif**; $p.\text{next}' = p.\text{next}$

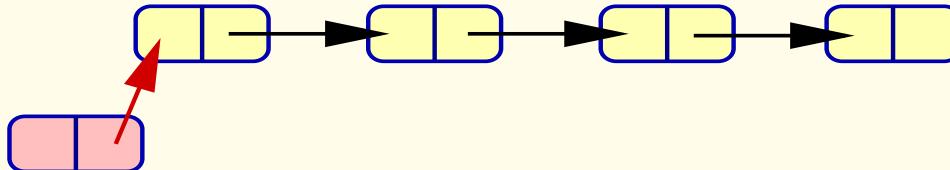
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Verification task: After insertion list remains sorted

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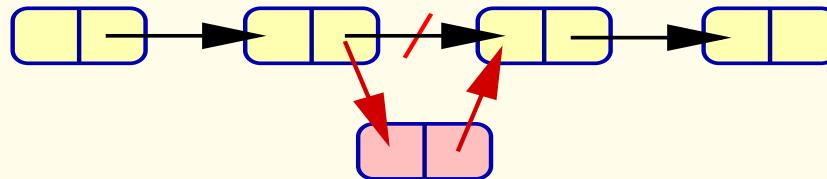
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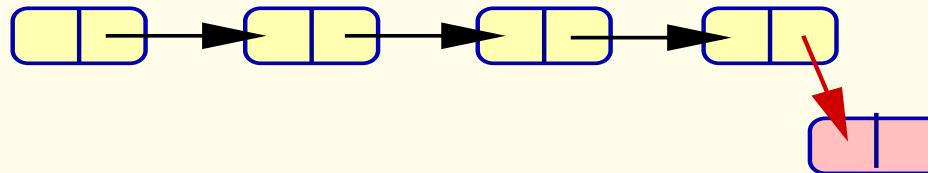
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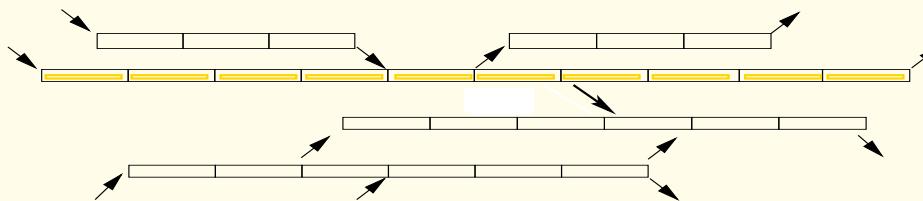
Verification task: After insertion list remains sorted

Applications

- Program verification
- More general verification problems

Model of a train controller

- Complex track topologies [Faber, Ihlemann, Jacobs, VS, IFM 2010]



Assumptions:

- No cycles
- in-degree (out-degree) of associated graph at most 2.

Data structures:

p_1 : trains

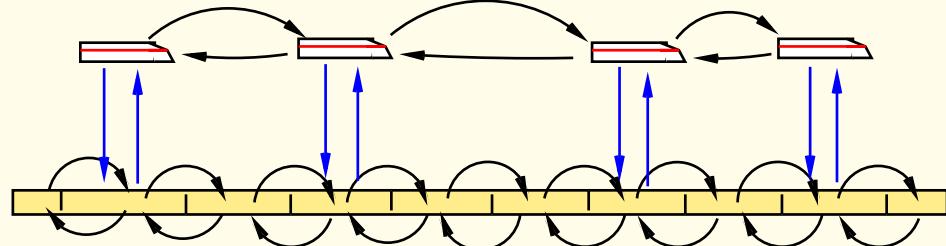
• 2-sorted pointers

p_2 : segments

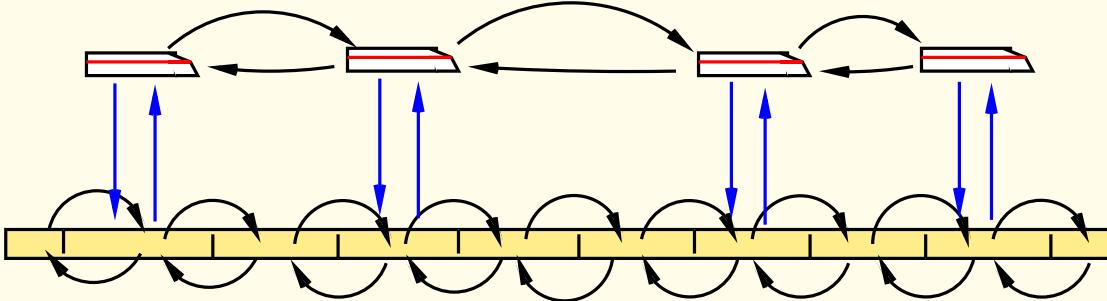
• scalar fields

speed: $p_1 \rightarrow \mathbb{R}$, $\text{id}_t: p_1 \rightarrow \mathbb{Z}$

MaxSpeed: $p_2 \rightarrow \mathbb{R}$, MaxBrakingDistance: $p_2 \rightarrow \mathbb{R}$, $\text{id}_s: p_1 \rightarrow \mathbb{Z}$



Incoming and outgoing trains



Example 1: Speed Update

$$\text{pos}(t) < \text{length}(\text{segm}(t)) - d \rightarrow 0 \leq \text{spd}'(t) \leq \text{lmax}(\text{segm}(t))$$

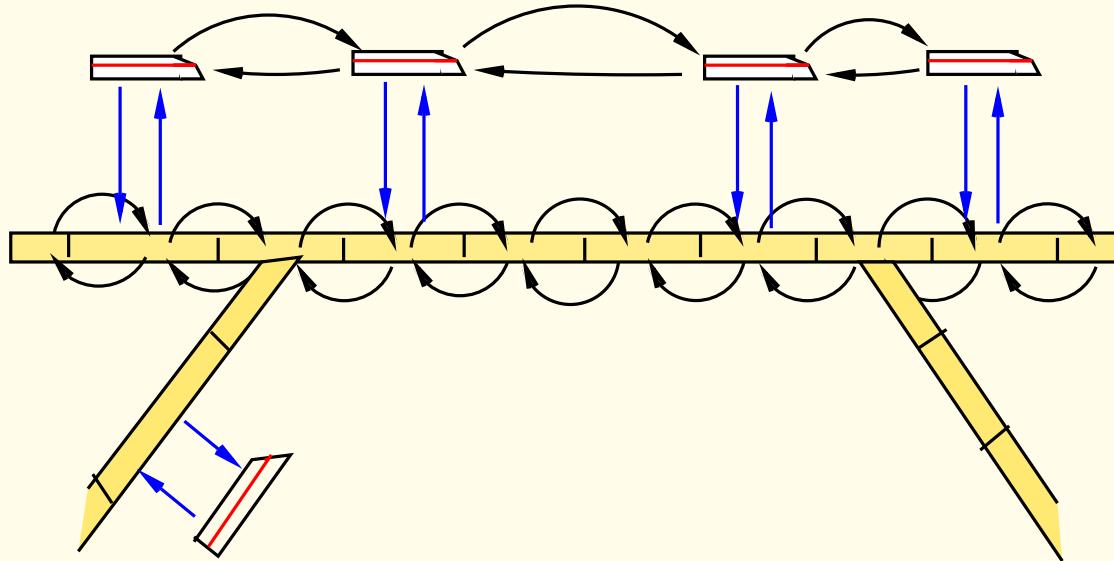
$$\text{pos}(t) \geq \text{length}(\text{segm}(t)) - d \wedge \text{alloc}(\text{next}_s(\text{segm}(t))) = \text{tid}(t)$$

$$\rightarrow 0 \leq \text{spd}'(t) \leq \min(\text{lmax}(\text{segm}(t)), \text{lmax}(\text{next}_s(\text{segm}(t))))$$

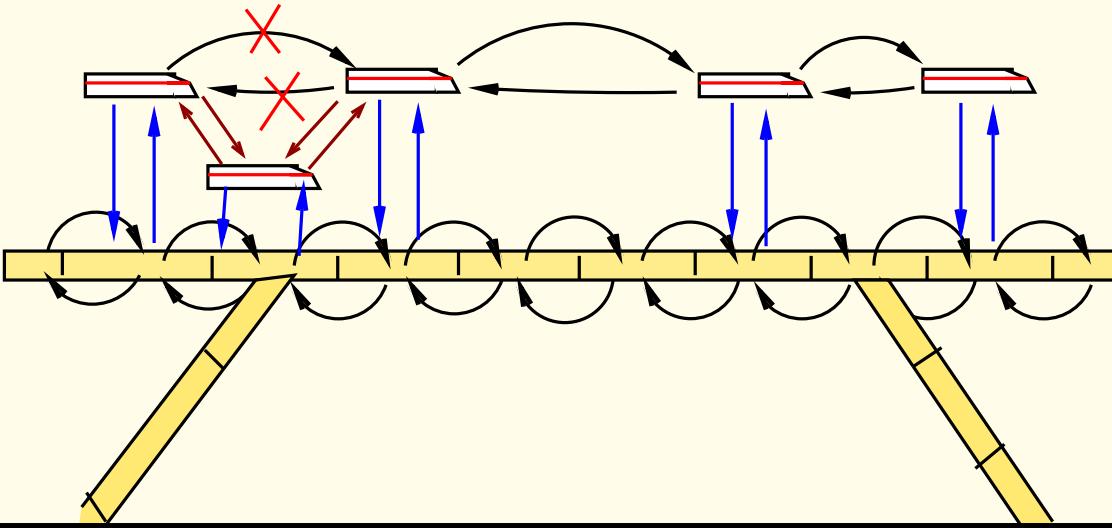
$$\text{pos}(t) \geq \text{length}(\text{segm}(t)) - d \wedge \text{alloc}(\text{next}_s(\text{segm}(t))) \neq \text{tid}(t)$$

$$\rightarrow \text{spd}'(t) = \max(\text{spd}(t) - \text{decmax}, 0)$$

Incoming and outgoing trains



Incoming and outgoing trains



Example 2: Enter Update (also updates for segm', spd', pos', train')

Assume: $s_1 \neq \text{null}_s$, $t_1 \neq \text{null}_t$, $\text{train}(s) \neq t_1$, $\text{alloc}(s_1) = \text{idt}(t_1)$

$t \neq t_1$, $\text{ids}(\text{segm}(t)) < \text{ids}(s_1)$, $\text{next}_t(t) = \text{null}_t$, $\text{alloc}(s_1) = \text{idt}(t_1) \rightarrow \text{next}'(t) = t_1 \wedge \text{next}'(t_1) = \text{null}_t$

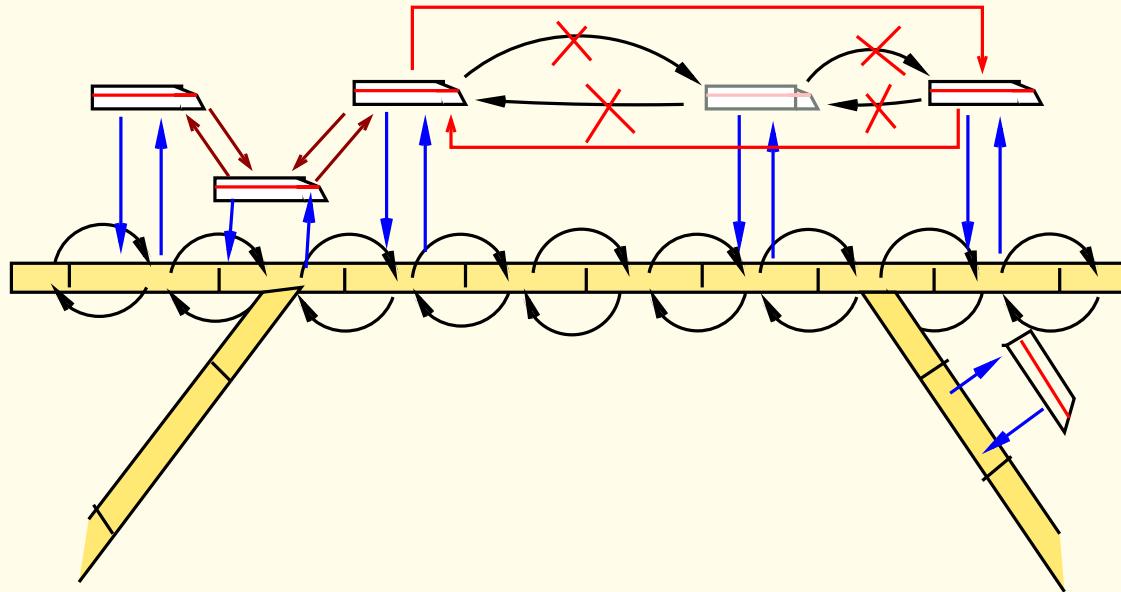
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$\rightarrow \text{next}'(t) = \text{next}_t(t)$

...

$t \neq t_1$, $\text{ids}(\text{segm}(t)) \geq \text{ids}(s_1) \rightarrow \text{next}'(t) = \text{next}_t(t)$

Incoming and outgoing trains



Safety property

Safety property we want to prove: no two trains ever occupy the same track segment:

$$(\text{Safe}) := \forall t_1, t_2 \text{ segm}(t_1) = \text{segm}(t_2) \rightarrow t_1 = t_2$$

We need to be able to efficiently reason in combinations of theories.

Example 2

Example: Does BUBBLESORT return
a sorted array?

```
int [] BUBBLESORT(int[] a) {  
    int i, j, t;  
    for (i := |a| - 1; i > 0; i := i - 1) {  
        for (j := 0; j < i; j := j + 1) {  
            if (a[j] > a[j + 1]) {t := a[j];  
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Example 2

Example: Does BUBBLESORT return
a sorted array?

$-1 \leq i < |a| \wedge$
 $\text{partitioned}(a, 0, i, i + 1, |a| - 1) \wedge$
 $\text{sorted}(a, i, |a| - 1)$

$-1 \leq i < |a| \wedge 0 \leq j \leq i \wedge$
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```

Generate verification conditions and prove that they are valid

Predicates:

- $\text{sorted}(a, l, u): \forall i, j (l \leq i \leq j \leq u \rightarrow a[i] \leq a[j])$
- $\text{partitioned}(a, l_1, u_1, l_2, u_2): \forall i, j (l_1 \leq i \leq u_1 \leq l_2 \leq j \leq u_2 \rightarrow a[i] \leq a[j])$

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```

$-1 \leq i < |a| \wedge$ $C_1(a)$
 $\text{partitioned}(a, 0, i, i + 1, |a| - 1) \wedge$
 $\text{sorted}(a, i, |a| - 1)$

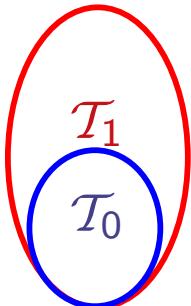
$-1 \leq i < |a| \wedge 0 \leq j \leq i \wedge$ $C_2(a)$
 $\text{partitioned}(a, 0, i, i + 1, |a| - 1) \wedge$
 $\text{sorted}(a, i, |a| - 1)$
 $\text{partitioned}(a, 0, j - 1, j, j)$

Generate verification conditions and prove that they are valid
 $C_2(a) \wedge \text{Update}(a, a') \rightarrow C_2(a')$

Motivation

Modular (i.e. black-box) composition of decision procedures is highly desirable – for saving time and resources.

Idea



Hierarchic Reasoning

\mathcal{T}_1 : Σ_1 -theory; $\mathcal{T}_0 \subseteq \mathcal{T}_1$ $\Sigma_0 \subset \Sigma_1$

\mathcal{T}_0 : Σ_0 -theory.

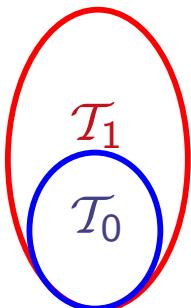
Example:

$f : \mathbb{R} \rightarrow \mathbb{R}$ mon.

\mathbb{R}

Can we use a prover for \mathcal{T}_0 as a blackbox to prove theorems in \mathcal{T}_1 ?

Idea



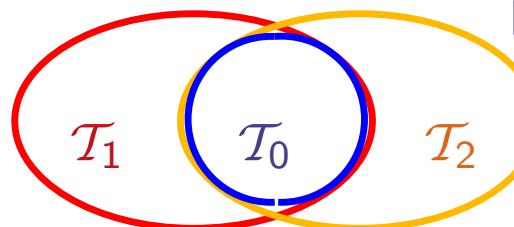
Hierarchic Reasoning

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Example:

$f : \mathbb{R} \rightarrow \mathbb{R}$ mon.
 \mathbb{R}

Can we use a prover for \mathcal{T}_0 as a blackbox to prove theorems in \mathcal{T}_1 ?



Modular Reasoning

\mathcal{T}_0 : Σ_0 -theory.
 \mathcal{T}_i : Σ_i -theory; $\mathcal{T}_0 \subseteq \mathcal{T}_i$ $\Sigma_0 \subseteq \Sigma_i$.

Example: **arrays**(\mathbb{Z}, \mathbb{R})

lists(\mathbb{R}) \cup **arrays**(\mathbb{Z}, \mathbb{R})

Can we use provers for $\mathcal{T}_1, \mathcal{T}_2$ as blackboxes to prove theorems in $\mathcal{T}_1 \cup \mathcal{T}_2$?

Which information needs to be exchanged between the provers?

Structure

- **Reasoning in standard theories**

Preliminaries: Logic, theories, models

Decidable logical theories and theory fragments

Tractability

- **Reasoning in combinations of theories**

disjoint signature

non-disjoint signature

- **Theory extensions**

- **Applications in Verification**

Reasoning in standard theories

- **Propositional logic and first-order logic** (reminder)
 - Syntax, Semantics, Entailment, Validity, Satisfiability
 - Theories, Models
 - Deduction in propositional logic (DPLL, resolution)
 - Deduction in first-order logic (the resolution principle)
- **Decidability and undecidability results**
 - Undecidability of FOL/Some decidable fragments
 - Deduction problems
- **Reasoning in numerical domains** (a crash course)

Reasoning in complex theories

- **Reasoning in combinations of theories**
 - combinations of theories with disjoint signatures
 - combinations of theories with non-disjoint signatures (main idea)
- **Theory extensions**
- **Applications**
 - Decision procedures for data structures:
 - arrays
 - pointer structures
 - sets (with cardinalities)
 - recursive data structures + recursive functions
 - Applications in verification

Literature

Logic:

Any book on logic (propositional and first-order logic)

- Schöning: Logik für Informatiker, Spektrum
- Fitting: First-Order Logic and Automated Theorem Proving, Springer
- Slides and suggested readings for the lecture “Automated Reasoning” (Saarbruecken, Summer Semester 2004), available online at:
<http://www.mpi-sb.mpg.de/~uwe/lehre/autreas/readings.html>

Literature

Decision procedures:

- Aaron Bradley and Zohar Manna: The Calculus of Computation: Decision Procedures with Applications to Verification. Springer, 2007.
- Daniel Kroening and Ofer Strichman: Decision Procedures: An algorithmic point of view. Springer, 2007.

Viorica Sofronie-Stokkermans: Reasoning in complex theories. Lecture notes; will be available online soon.

Semesterapparat at the library

Organization

Organization

2h Lecture + 2h Exercises

Time:

Mondays: Lecture 14:00-16:00, Room C206

Exercise: 16:00-18:00, Room B017 (Markus Bender)

website of the lecture:

<http://www.uni-koblenz.de/~sofronie/lecture-dp-ws-2012/>

Homework

- will be available online after the lecture
(at latest on Tuesday)
- due on Friday, 17:00.

Exam

Exam:

- form (oral/written): to be decided