Universität Koblenz-Landau

FB 4 Informatik

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Exercises for "Decision Procedures for Verification"
Exercise sheet 7

Exercise 7.1: *(2 P)*
Assume $S \succ P \succ Q \succ R$. Let N be the following set of clauses:

$$\begin{array}{ll} (1) & \neg Q \vee P \vee R \\ (2) & \neg R \vee P \\ (3) & Q \vee S \vee \neg P \\ (4) & \neg Q \vee \neg S \end{array}$$

Give the definition of redundancy of a clause w.r.t. a set of clauses.

- Is the clause $\neg Q \vee P \vee S$ redundant w.r.t. the set N above?
- Is the clause $\neg Q \vee P$ redundant w.r.t. the set N above?

Justify your answer.

Exercise 7.2: *(2 P)*
Assume $U \succ S \succ P \succ Q \succ R$. Let N be the following set of clauses:

$$\begin{array}{ll} (1) & \neg Q \vee P \vee R \\ (2) & \neg R \vee P \\ (3) & \neg Q \vee P \vee S \\ (4) & Q \vee S \vee \neg P \\ (5) & \neg Q \vee \neg S \end{array}$$

- Is the clause $\neg Q \vee P \vee S \vee U$ redundant w.r.t. the set consisting of the clauses (1), (2), (4) and (5)?

Justify your answer.

Exercise 7.3: *(2 P)*
Let ϕ be the following (ground) formula:

$$f(f(c)) \approx f(c) \wedge f(f(c)) \approx f(d) \wedge d \not\approx f(c).$$

(1) Compute $FLAT(\phi)$ (the formula obtained by recursively replacing, in a bottom-up fashion, any term of the form $f(c')$, where c' is a constant, with a new constant).

(2) Compute $FC(\phi)$ (the set of functional consistency axioms associated with the flattening above):

$$FC(\phi) = \{c_1 \approx c_2 \rightarrow d_1 \approx d_2 \mid d_i \text{ is introduced as an abbreviation for } f(c_i)\}.$$

(3) Check whether $FLAT(\phi) \wedge FC(\phi)$ is satisfiable.

(4) Is ϕ is satisfiable? Justify your answer.

Supplementary exercises

Exercise 7.4: (*A P*)
set of Horn clauses in first-order logic without equality is called *superficial* if for every clause

$$A_1 \wedge \dots \wedge A_n \rightarrow A \qquad \text{(alternatively written also:} \neg A_1 \vee \dots \vee \neg A_n \vee A)$$

in H (where $A_1, \dots, A_n, A$ are atoms), for every term t occurring in A there exists $j \in \{1, \dots, n\}$ such that t occurs in A_j.

Let H be a superficial set of Horn clauses and let G be a conjunction of ground literals.

Prove that $H \cup G$ is satisfiable if and only if $H[G] \cup G$ is satisfiable,
where $H[G]$ is the set of all ground instances of H which contain only ground terms occurring in G or which already occur in H.

Exercise 7.5: (*2 P*)

Let F be a closed first-order formula with equality over a signature $\Sigma = (\Omega, \Pi)$. Let $\sim \notin \Omega$ be a new binary relation symbol (written as an infix operator). Let the set $Eq(\Sigma)$ contain the formulas

$$\forall x\, (x \sim x)$$
$$\forall x, y\, (x \sim y \rightarrow y \sim x)$$
$$\forall x, y, z\, (x \sim y \wedge y \sim z \rightarrow x \sim z)$$

and for every $f/n \in \Omega$ the formula

$$\forall x_1, \dots, x_n, y_1, \dots, y_n\, (x_1 \sim y_1 \wedge \dots \wedge x_n \sim y_n \rightarrow f(x_1, \dots, x_n) \sim f(y_1, \dots, y_n))$$

and for every $p/n \in \Pi$ the formula

$$\forall x_1, \dots, x_n, y_1, \dots, y_n\, (x_1 \sim y_1 \wedge \dots \wedge x_n \sim y_n \wedge p(x_1, \dots, x_n) \rightarrow p(y_1, \dots, y_n)).$$

Let $\tilde{F}$ be the formula that one obtains from F if every occurrence of the equality symbol $\approx$ is replaced by the relation symbol $\sim$.

(a) *Definition.* A binary relation $\sim$ on the support of a Σ-algebra satisfying all properties in $Eq(\Sigma)$ is called a congruence relation.

Let $\mathcal{A}$ be a model of $\tilde{F} \cup Eq(\Sigma)$. Show that the interpretation $\sim_{\mathcal{A}}$ of $\sim$ in $\mathcal{A}$ is a congruence relation. (It is enough if you prove *one* of the properties of congruence relations, say symmetry; the other properties are proved analogously.)

(b) Let $\mathcal{A}$ be a model of $\tilde{F} \cup Eq(\Sigma)$. Use the congruence relation $\sim_{\mathcal{A}}$ to construct a model of F and prove that it is a model of F.

Hint: Construct the quotient $\hat{\mathcal{A}} = \mathcal{A}/\sim_{\mathcal{A}}$ of $\mathcal{A}$ as follows:

- $U_{\hat{\mathcal{A}}} = \{[x] \mid x \in U_{\mathcal{A}}\}$, where $[x] = \{y \in U_{\mathcal{A}} \mid y \sim_{\mathcal{A}} x\}$ is the equivalence class of x w.r.t. $\sim_{\mathcal{A}}$.
- for every $f/n \in \Omega$, define $f_{\hat{\mathcal{A}}} : U^n_{\hat{\mathcal{A}}} \to U_{\hat{\mathcal{A}}}$ by $f_{\hat{\mathcal{A}}}([x_1], \dots, [x_n]) = [f_{\mathcal{A}}(x_1, \dots, x_n)]$.
- for every $p/m \in \Pi$, define $p_{\hat{\mathcal{A}}} \subseteq U^m_{\hat{\mathcal{A}}}$ by: $([x_1], \dots, [x_n]) \in p_{\hat{\mathcal{A}}}$ iff $(x_1, \dots, x_n) \in p_{\mathcal{A}}$.

(c) Prove that a formula F is satisfiable if and only if $Eq(\Sigma) \cup \{\tilde{F}\}$ is satisfiable.

Please submit your solution until Monday, December 9, 2013 at 16:00. Joint solutions prepared by up to three persons are allowed. Please do not forget to write your name on your solution.

Submission possibilities:

- By e-mail to `sofronie@uni-koblenz.de` with the keyword “Homework DP” in the subject.
- Put it in the box in front of Room B 222.