Decision Procedures in Verification

Decision Procedures (1)

5.12.2013

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Until now:

Syntax (one-sorted signatures vs. many-sorted signatures)

Semantics

Structures (also many-sorted)
Models, Validity, and Satisfiability
Entailment and Equivalence

Theories (Syntactic vs. Semantics view)

Algorithmic Problems

Decidability/Undecidability
Methods: Resolution (Soundness, refutational completeness, refinements)
Consequences: Compactness of FOL; The Löwenheim-Skolem Theorem; Craig interpolation

Decidable subclasses of FOL

The Bernays-Schönfinkel class
(definition; decidability;tractable fragment: Horn clauses)
The Ackermann class
The monadic class

The Monadic Class

Monadic first-order logic (MFO) is FOL (without equality) over purely relational signatures $\Sigma = (\Omega, \Pi)$, where $\Omega = \emptyset$, and every $p \in \Pi$ has arity 1.

Abstract syntax:

$$\Phi := \top \mid P(x) \mid \Phi_1 \wedge \Phi_2 \mid \neg\Phi \mid \forall x \Phi$$

– All predicates unary

– No functions

– No restrictions on the formulae or on the quantifier prefix

The Monadic Class

MFO Abstract syntax: $\Phi := \top \mid P(x) \mid \Phi_1 \wedge \Phi_2 \mid \neg\Phi \mid \forall x \Phi$

Theorem (Finite model theorem for MFO). If Φ is a satisfiable MFO formula with k predicate symbols then Φ has a model where the domain is a subset of $\{0, 1\}^k$.

Idea. Let Φ be a MFO formula with k predicate symbols.

Let $\mathcal{A} = (U_\mathcal{A}, \{p_\mathcal{A}\}_{p \in \Pi})$ be a Σ-algebra. The only way to distinguish the elements of $U_\mathcal{A}$ is by the atomic formulae $p(x)$, $p \in \Pi$.

- the elements which $a \in U_\mathcal{A}$ which belong to the same $p_\mathcal{A}$'s, $p \in \Pi$ can be collapsed into one single element.
- if $\Pi = \{p^1, \dots, p^k\}$ then what remains is a *finite structure* with at most 2^k elements.
- the truth value of a formula: computed by evaluating all subformulae.

The Monadic Class

Theorem (Finite model theorem for MFO). If Φ is a satisfiable MFO formula with k predicate symbols then Φ has a model where the domain is a subset of $\{0,1\}^k$.

Proof: Let $\mathcal{B} = (\{0,1\}^k, \{p^1_\mathcal{B}, \dots, p^k_\mathcal{B}\})$, where $p^i_\mathcal{B} = \{(b_1, \dots, b_k) \mid b_i = 1\}$.

Let $\mathcal{A} = (U_\mathcal{A}, \{p^1_\mathcal{A}, \dots, p^k_\mathcal{A}\})$, $\beta : X \to U_\mathcal{A}$ be such that $(\mathcal{A}, \beta) \models \Phi$.
We construct a model for Φ with cardinality at most 2^k as follows:

- Let $h : \mathcal{A} \to \mathcal{B}$ be defined for all $a \in U_\mathcal{A}$ by:

  $$h(a) = (b_1, \dots, b_k) \text{ where } b_i = 1 \text{ if } a \in p^i_\mathcal{A} \text{ and } 0 \text{ otherwise.}$$

  Then $a \in p^i_\mathcal{A}$ iff $h(a) \in p^i_\mathcal{B}$ for all $a \in U_\mathcal{A}$ and all $i = 1, \dots, k$.
- Let $\mathcal{B}' = (\{0,1\}^k \cap h(U_\mathcal{A}), \{p^1_\mathcal{B} \cap h(U_\mathcal{A}), \dots, p^k_\mathcal{B} \cap h(U_\mathcal{A})\})$.
- We show that $(\mathcal{B}', \beta \circ h) \models \Phi$. Structural induction

The Monadic Class

To show:
$(\mathcal{A}(\beta)(\Phi) = \mathcal{B}'(\beta \circ h)(\Phi)$.

Induction on the structure of Φ

Induction base: Show that claim is true for all atomic formulae

- $\Phi = \top$ OK

- $\Phi = p^i(x)$.

  Then the following are equivalent:

  (1) $(\mathcal{A}, \beta) \models \Phi$
  (2) $\beta(x) \in p^i_{\mathcal{A}}$ (definition)
  (3) $h(\beta(x)) \in p^i_{\mathcal{B}}$ (definition of h and of $p^i_{\mathcal{B}}$)
  (4) $(\mathcal{B}', \beta \circ h) \models \Phi$ (definition)

The Monadic Class

Induction on the structure of Φ

Let Φ be a formula which is not atomic.

Assume statement holds for the (direct) subformulae of Φ. Prove that it holds for Φ.

- $\Phi = \Phi_1 \wedge \Phi_2$

  Assume $(\mathcal{A}, \beta) \models \Phi$. Then $(\mathcal{A}, \beta) \models \Phi_i$, $i = 1, 2$.

  By induction hypothesis, $(\mathcal{B}', \beta \circ h) \models \Phi_i$, $i = 1, 2$.

  Thus, $(\mathcal{B}', \beta \circ h) \models \Phi = \Phi_1 \wedge \Phi_2$

  The converse can be proved similarly.

- $\Phi = \neg\Phi_1$

  The following are equivalent:

  (1) $(\mathcal{A}, \beta) \models \Phi = \neg\Phi_1$.

  (2) $\mathcal{A}(\beta)(\Phi_1) = 0$

  (3) $\mathcal{B}'(\beta \circ h)(\Phi_1) = 0$ (induction hypothesis)

  (4) $(\mathcal{B}', \beta \circ h) \models \Phi = \neg\Phi_1$

The Monadic Class

- $\Phi = \forall x \Phi_1(x)$.

  Then the following are equivalent:

  (1) $(\mathcal{A}, \beta) \models \Phi$

  (2) $\mathcal{A}(\beta[x \mapsto a])(\Phi_1) = 1$ for all $a \in U_{\mathcal{A}}$

  (3) $\mathcal{B}'(\beta[x \mapsto a] \circ h)(\Phi_1) = 1$ for all $a \in U_{\mathcal{A}}$ (ind. hyp)

  (4) $\mathcal{B}'(\beta \circ h[x \mapsto b])(\Phi_1) = 1$ for all $b \in \{0, 1\}^k \cap h(A)$

  (5) $(\mathcal{B}', \beta \circ h) \models \Phi$

The Monadic Class

Resolution-based decision procedure for the Monadic Class (and for several other classes):

William H. Joyner Jr.
Resolution Strategies as Decision Procedures.
J. ACM 23(3): 398-417 (1976)

Idea:

- Use orderings to restrict the possible inferences
- Identify a class of clauses (with terms of bounded depth) which contains the type of clauses generated from the respective fragment and is closed under ordered resolution (+ red. elim. criteria)
- Show that a saturation of the clauses can be obtained in finite time

The Monadic Class

Resolution-based decision procedure for the Monadic Class:

$$\Phi: \quad \forall \overline{x}_1 \exists \overline{y}_1 \ldots \forall \overline{x}_k \exists \overline{y}_k (....p^s(x_i)......p^l(y_i)...)$$

$$\mapsto \quad \forall \overline{x}_1 \ldots \forall \overline{x}_k (...p^s(x_i)...p^l(f_{\mathrm{sk}}(\overline{x}_1, \ldots, \overline{x}_i)...)$$

Consider the class MON of clauses with the following properties:

- no literal of heigth greater than 2 appears
- each variable-disjoint partition has at most $n = \sum_{i=1} |\overline{x}_i|$ variables (can order the variables as $x_1, \ldots, x_n$)
- the variables of each non-ground block can occur either in atoms $p(x_i)$ or in atoms $P(f_{\mathrm{sk}}(x_1, \ldots, x_t))$, $0 \leq t \leq n$

It can be shown that this class contains all CNF's of formulae in the monadic class and is closed under ordered resolution.

3.2 Deduction problems

Satisfiability w.r.t. a theory

Satisfiability w.r.t. a theory

Example

Let $\Sigma = (\{e/0, */2, i/1\}, \emptyset)$

Let $\mathcal{F}$ consist of all (universally quantified) group axioms:

$$\begin{array}{ll} \forall x, y, z & x * (y * z) \approx (x * y) * z \\ \forall x & x * i(x) \approx e \quad \wedge \quad i(x) * x \approx e \\ \forall x & x * e \approx x \quad \wedge \quad e * x \approx x \end{array}$$

Question: Is $\forall x, y (x * y = y * x)$ entailed by $\mathcal{F}$?

Satisfiability w.r.t. a theory

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Question: Is $\forall x, y (x * y = y * x)$ entailed by $\mathcal{F}$?

Alternative question:
Is $\forall x, y (x * y = y * x)$ true in the class of all groups?

Logical theories

Syntactic view

first-order theory: given by a set $\mathcal{F}$ of (closed) first-order Σ-formulae.

the models of $\mathcal{F}$: $\text{Mod}(\mathcal{F}) = \{\mathcal{A} \in \Sigma\text{-alg} \mid \mathcal{A} \models G, \text{ for all } G \text{ in } \mathcal{F}\}$

Semantic view

given a class $\mathcal{M}$ of Σ-algebras

the first-order theory of $\mathcal{M}$: $\text{Th}(\mathcal{M}) = \{G \in F_\Sigma(X) \text{ closed} \mid \mathcal{M} \models G\}$

Decidable theories

Let $\Sigma = (\Omega, \Pi)$ be a signature.

$\mathcal{M}$: class of Σ-algebras. $\mathcal{T} = \text{Th}(\mathcal{M})$ is decidable

iff

there is an algorithm which, for every closed first-order formula ϕ, can decide (after a finite number of steps) whether ϕ is in $\mathcal{T}$ or not.

$\mathcal{F}$: class of (closed) first-order formulae.

The theory $\mathcal{T} = \text{Th}(\text{Mod}(\mathcal{F}))$ is decidable

iff

there is an algorithm which, for every closed first-order formula ϕ, can decide (in finite time) whether $\mathcal{F} \models \phi$ or not.

Examples

Undecidable theories

- $\mathrm{Th}((\mathbb{Z}, \{0, 1, +, *\}, \{\leq\}))$
- $\mathrm{Th}(\Sigma\text{-alg})$

Peano arithmetic

Peano axioms:

$\forall x\, \neg(x + 1 \approx 0)$ (zero)

$\forall x \forall y\, (x + 1 \approx y + 1 \rightarrow x \approx y$ (successor)

$F[0] \wedge (\forall x\, (F[x] \rightarrow F[x + 1]) \rightarrow \forall x F[x])$ (induction)

$\forall x\, (x + 0 \approx x)$ (plus zero)

$\forall x, y\, (x + (y + 1) \approx (x + y) + 1)$ (plus successor)

$\forall x, y\, (x * 0 \approx 0)$ (times 0)

$\forall x, y\, (x * (y + 1) \approx x * y + x)$ (times successor)

$3 * y + 5 > 2 * y$ expressed as $\exists z (z \neq 0 \wedge 3 * y + 5 \approx 2 * y + z)$

Intended interpretation: $(\mathbb{N}, \{0, 1, +, *\}, \{\approx, \leq\})$
(does not capture true arithmetic by Goedel's incompleteness theorem)

Examples

In order to obtain decidability results:

- Restrict the signature
- Enrich axioms
- Look at certain fragments

Examples

In order to obtain decidability results:

- Restrict the signature
- Enrich axioms
- Look at certain fragments

Decidable theories

- Presburger arithmetic decidable in 3EXPTIME [Presburger'29]
  Signature: $(\{0, 1, +\}, \{\approx, \leq\})$ (no $*$)

  Axioms { (zero), (successor), (induction), (plus zero), (plus successor) }

- Th($\mathbb{Z}_+$) $\mathbb{Z}_+ = (\mathbb{Z}, 0, s, +, \leq)$ the standard interpretation of integers.

Examples

In order to obtain decidability results:

- Restrict the signature
- Enrich axioms
- Look at certain fragments

Decidable theories

- The theory of real numbers (with addition and multiplication) is decidable in 2EXPTIME [Tarski'30]

Examples

In order to obtain decidability results:

- Restrict the signature
- Enrich axioms
- Look at certain fragments

Problems

$\mathcal{T}$: first-order theory in signature Σ; $\mathcal{L}$ class of (closed) Σ-formulae

Given ϕ in $\mathcal{L}$, is it the case that $\mathcal{T} \models \phi$?

Common restrictions on $\mathcal{L}$

	Pred $= \emptyset$	$\{\phi \in \mathcal{L} \mid \mathcal{T} \models \phi\}$
$\mathcal{L}=\{\forall x A(x) \mid A \text{ atomic}\}$	word problem	
$\mathcal{L}=\{\forall x (A_1 \wedge \ldots \wedge A_n \rightarrow B) \mid A_i, B \text{ atomic}\}$	uniform word problem	$\mathrm{Th}_{\forall \mathrm{Horn}}$
$\mathcal{L}=\{\forall x C(x) \mid C(x) \text{ clause}\}$	clausal validity problem	$\mathrm{Th}_{\forall, \mathrm{cl}}$
$\mathcal{L}=\{\forall x \phi(x) \mid \phi(x) \text{ unquantified}\}$	universal validity problem	$\mathrm{Th}_{\forall}$
$\mathcal{L}=\{\exists x A_1 \wedge \ldots \wedge A_n \mid A_i \text{ atomic}\}$	unification problem	$\mathrm{Th}_{\exists}$
$\mathcal{L}=\{\forall x \exists x A_1 \wedge \ldots \wedge A_n \mid A_i \text{ atomic}\}$	unification with constants	$\mathrm{Th}_{\forall \exists}$

$\mathcal{T}$-validity vs. $\mathcal{T}$-satisfiability

$\mathcal{T}$-validity: Let $\mathcal{T}$ be a first-order theory in signature Σ
Let $\mathcal{L}$ be a class of (closed) Σ-formulae

Given ϕ in $\mathcal{L}$, is it the case that $\mathcal{T} \models \phi$?

Remark: $\mathcal{T} \models \phi$ iff $\mathcal{T} \cup \neg\phi$ unsatisfiable

Every $\mathcal{T}$-validity problem has a dual $\mathcal{T}$-satisfiability problem:

$\mathcal{T}$-satisfiability: Let $\mathcal{T}$ be a first-order theory in signature Σ
Let $\mathcal{L}$ be a class of (closed) Σ-formulae

$$\neg\mathcal{L} = \{\neg\phi \mid \phi \in \mathcal{L}\}$$

Given ψ in $\neg\mathcal{L}$, is it the case that $\mathcal{T} \cup \psi$ is satisfiable?

$\mathcal{T}$-validity vs. $\mathcal{T}$-satisfiability

Common restrictions on $\mathcal{L}$ / $\neg\mathcal{L}$

$\mathcal{L}$	$\neg\mathcal{L}$
$\{\forall x A(x) \mid A \text{ atomic}\}$	$\{\exists x \neg A(x) \mid A \text{ atomic}\}$
$\{\forall x (A_1 \wedge \ldots \wedge A_n \rightarrow B) \mid A_i, B \text{ atomic}\}$	$\{\exists x (A_1 \wedge \ldots \wedge A_n \wedge \neg B) \mid A_i, B \text{ atomic}\}$
$\{\forall x \bigvee L_i \mid L_i \text{ literals}\}$	$\{\exists x \bigwedge L'_i \mid L'_i \text{ literals}\}$
$\{\forall x \phi(x) \mid \phi(x) \text{ unquantified}\}$	$\{\exists x \phi'(x) \mid \phi'(x) \text{ unquantified}\}$

validity problem for universal formulae

ground satisfiability problem

$\mathcal{T}$-validity vs. $\mathcal{T}$-satisfiability

Common restrictions on $\mathcal{L}$ / $\neg\mathcal{L}$

$\mathcal{L}$	$\neg\mathcal{L}$
$\{\forall x A(x) \mid A \text{ atomic}\}$	$\{\exists x \neg A(x) \mid A \text{ atomic}\}$
$\{\forall x (A_1 \wedge \ldots \wedge A_n \rightarrow B) \mid A_i, B \text{ atomic}\}$	$\{\exists x (A_1 \wedge \ldots \wedge A_n \wedge \neg B) \mid A_i, B \text{ atomic}\}$
$\{\forall x \bigvee L_i \mid L_i \text{ literals}\}$	$\{\exists x \bigwedge L'_i \mid L'_i \text{ literals}\}$
$\{\forall x \phi(x) \mid \phi(x) \text{ unquantified}\}$	$\{\exists x \phi'(x) \mid \phi'(x) \text{ unquantified}\}$

validity problem for universal formulae — ground satisfiability problem

In what follows we will focus on the problem of checking the satisfiability of conjunctions of ground literals

$\mathcal{T}$-validity vs. $\mathcal{T}$-satisfiability

$\mathcal{T} \models \forall x A(x)$	iff	$\mathcal{T} \cup \exists x \neg A(x)$ unsatisfiable
$\mathcal{T} \models \forall x (A_1 \wedge \cdots \wedge A_n \rightarrow B)$	iff	$\mathcal{T} \cup \exists x (A_1 \wedge \cdots \wedge A_n \wedge \neg B)$ unsatisfiable
$\mathcal{T} \models \forall x (\bigvee_{i=1}^{n} A_i \vee \bigvee_{j=1}^{m} \neg B_j)$	iff	$\mathcal{T} \cup \exists x (\neg A_1 \wedge \cdots \wedge \neg A_n \wedge B_1 \wedge \cdots \wedge B_m)$ unsatisfiable

$\mathcal{T}$-satisfiability vs. Constraint Solving

The field of Constraint Solving also deals with satisfiability problems

But be careful:

- in Constraint Solving one is interested if a formula is satisfiable in a given, fixed model of $\mathcal{T}$.
- in $\mathcal{T}$-satisfiability one is interested if a formula is satisfiable in any model of $\mathcal{T}$ at all.

3.3. Theory of Uninterpreted Function Symbols

Why?

- Reasoning about equalities is important in automated reasoning
- Applications to program verification
  (approximation: abstract from additional properties)

Application: Compiler Validation

Example: prove equivalence of source and target program

```
1:  y := 1
2:  if z = x*x*x
3:    then y := x*x + y
4:  endif
```

```
1: y := 1
2: R1 := x*x
3: R2 := R1*x
4: jmpNE(z,R2,6)
5: y := R1+1
```

To prove: (indexes refer to values at line numbers)

$$y_1 \approx 1 \land [(z_0 \approx x_0 * x_0 * x_0 \land y_3 \approx x_0 * x_0 + y_1) \lor (z_0 \not\approx x_0 * x_0 * x_0 \land y_3 \approx y_1)] \land$$

$$y'_1 \approx 1 \land R1_2 \approx x'_0 * x'_0 \land R2_3 \approx R1_2 * x'_0 \land$$

$$\land [(z'_0 \approx R2_3 \land y'_5 \approx R1_2 + 1) \lor (z'_0 \neq R2_3 \land y'_5 \approx y'_1)] \land$$

$$x_0 \approx x'_0 \land y_0 \approx y'_0 \land z_0 \approx z'_0 \quad \Longrightarrow \quad x_0 \approx x'_0 \land y_3 \approx y'_5 \land z_0 \approx z'_0$$

Possibilities for checking it

(1) **Abstraction.**
Consider $*$ to be a "free" function symbol (forget its properties). Test it property can be proved in this approximation. If so, then we know that implication holds also under the normal interpretation of $*$.

(2) **Reasoning about formulae in fragments of arithmetic.**

Uninterpreted function symbols

Let $\Sigma = (\Omega, \Pi)$ be arbitrary

Let $\mathcal{M} = \Sigma$-alg be the class of all Σ-structures

The theory of uninterpreted function symbols is Th(Σ-alg) the family of all first-order formulae which are true in all Σ-algebras.

in general undecidable

Decidable fragment:

e.g. the class $\text{Th}_\forall(\Sigma\text{-alg})$ of all universal formulae which are true in all Σ-algebras.

Uninterpreted function symbols

Assume $\Pi = \emptyset$ (and $\approx$ is the only predicate)

In this case we denote the theory of uninterpreted function symbols by $UIF(\Sigma)$ (or UIF when the signature is clear from the context).

This theory is sometimes called the theory of free functions and denoted $\text{Free}(\Sigma)$

Uninterpreted function symbols

Theorem 3.3.1

The following are equivalent:

(1) testing validity of universal formulae w.r.t. UIF is decidable

(2) testing validity of (universally quantified) clauses w.r.t. UIF is decidable

Proof: Follows from the fact that any universal formula is equivalent to a conjunction of (universally quantified) clauses.

Solution 1

Task:

Check if $UIF \models \forall \overline{x}(s_1(\overline{x}) \approx t_1(\overline{x}) \wedge \cdots \wedge s_k(\overline{x}) \approx t_k(\overline{x}) \rightarrow \bigvee_{j=1}^{m} s'_j(\overline{x}) \approx t'_j t(\overline{x}))$

Solution 1:

The following are equivalent:

(1) $(\bigwedge_i s_i \approx t_i) \rightarrow \bigvee_j s'_j \approx t'_j$ is valid

(2) $Eq(\sim) \wedge \text{Con}(f) \wedge (\bigwedge_i s_i \sim t_i) \wedge (\bigwedge_j s'_j \not\sim t'_j)$ is unsatisfiable.

where $Eq(\sim) : \text{Refl}(\sim) \wedge \text{Sim}(\sim) \wedge \text{Trans}(\sim)$

$\text{Con}(f) : \forall x_1, \dots, x_n, y_1, \dots, y_n (\bigwedge x_i \sim y_i \rightarrow f(x_1, \dots, x_n) \sim f(y_1, \dots, y_n))$

Resolution: inferences between transitivity axioms – nontermination

Solution 2

Task:

Check if $UIF \models \forall \overline{x}(s_1(\overline{x}) \approx t_1(\overline{x}) \wedge \cdots \wedge s_k(\overline{x}) \approx t_k(\overline{x}) \rightarrow \bigvee_{j=1}^{m} s'_j(\overline{x}) \approx t'_j(\overline{x}))$

Solution 2: Ackermann's reduction.

Flatten the formula (replace, bottom-up, $f(c)$ with a new constant c_f
$\phi \mapsto FLAT(\phi)$

Theorem 3.3.2: The following are equivalent:

(1) $(\bigwedge_i s_i(\overline{c}) \approx t_i(\overline{c})) \wedge \bigwedge_j s'_j(\overline{c}) \not\approx t'_j(\overline{c})$ is satisfiable
(2) $FC \wedge FLAT[(\bigwedge_i s_i(\overline{c}) \approx t_i(\overline{c})) \wedge \bigwedge_j s'_j(\overline{c}) \not\approx t'_j(\overline{c})]$ is satisfiable

where $FC = \{c_1{=}d_1, \dots c_n{=}d_n \rightarrow c_f{=}d_f \mid$ whenever $f(c_1, \dots, c_n)$ was renamed to c_f
$f(d_1, \dots, d_n)$ was renamed to $d_f\}$

Note: The problem is decidable in PTIME (see next pages)

Problem: Naive handling of transitivity/congruence axiom $\mapsto O(n^3)$

Goal: Give a faster algorithm

Example

The following are equivalent:

(1) $C := f(a, b) \approx a \wedge f(f(a, b), b) \not\approx a$

(2) $FC \wedge FLAT[C]$, where:

$FLAT[f(a, b) \approx a \wedge f(f(a, b), b) \not\approx a]$ is computed by introducing new constants renaming terms starting with f and then replacing in C the terms with the constants:

- $FLAT[\underbrace{f(a, b)}_{a_1} \approx a \wedge \underbrace{f(\underbrace{f(a, b)}_{a_1}, b)}_{a_2} \not\approx a] := a_1 \approx a \wedge a_2 \not\approx a$
- $FC := (a \approx a_1 \rightarrow a_1 \approx a_2)$

$f(a, b) = a_1$

$f(a_1, b) = a_2$

Thus, the following are equivalent:

(1) $C := f(a, b) \approx a \wedge f(f(a, b), b) \not\approx a$

(2) $\underbrace{(a \approx a_1 \rightarrow a_1 \approx a_2)}_{FC} \wedge \underbrace{a_1 \approx a \wedge a_2 \not\approx a}_{FLAT[C]}$

Solution 3

Task:

Check if $UIF \models \forall \overline{x}(s_1(\overline{x}) \approx t_1(\overline{x}) \wedge \cdots \wedge s_k(\overline{x}) \approx t_k(\overline{x}) \rightarrow \bigvee_{j=1}^{m} s'_j(\overline{x}) \approx t'_j(\overline{x}))$

i.e. if $(s_1(\overline{c}) \approx t_1(\overline{c}) \wedge \cdots \wedge s_k(\overline{c}) \approx t_k(\overline{c}) \wedge \bigwedge_j s'_j(\overline{c}) \not\approx t'_j(\overline{c}))$ unsatisfiable.

Solution 3

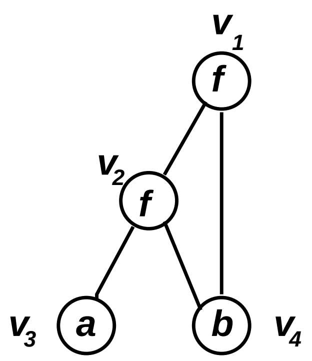
Task:

Check if $(s_1(\overline{c}) \approx t_1(\overline{c}) \wedge \cdots \wedge s_k(\overline{c}) \approx t_k(\overline{c}) \wedge \bigwedge_k s'_k(\overline{c}) \not\approx t'_k(\overline{c}))$ unsatisfiable.

Solution 3 [Downey-Sethi, Tarjan'76; Nelson-Oppen'80]

represent the terms occurring in the problem as DAG's

Example: Check whether $f(f(a, b), b) \approx a$ is a consequence of $f(a, b) \approx a$.



$$
\begin{aligned}
v_1 &: \quad f(f(a, b), b) \\
v_2 &: \quad f(a, b) \\
v_3 &: \quad a \\
v_4 &: \quad b
\end{aligned}
$$

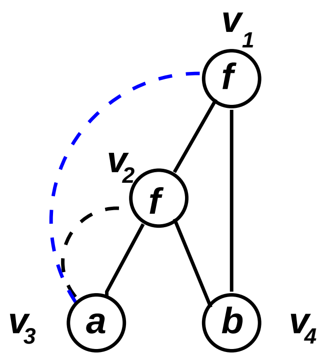
Solution 3

Task: Check if $(s_1(\overline{c}) \approx t_1(\overline{c}) \land \cdots \land s_k(\overline{c}) \approx t_k(\overline{c}) \land s(\overline{c}) \not\approx t(\overline{c}))$ unsatisfiable.

Solution 3 [Downey-Sethi, Tarjan'76; Nelson-Oppen'80]

- represent the terms occurring in the problem as DAG's
- represent premise equalities by a relation on the vertices of the DAG

Example: Check whether $f(f(a, b), b) \approx a$ is a consequence of $f(a, b) \approx a$.



v_1 : $f(f(a, b), b)$

v_2 : $f(a, b)$

v_3 : a

v_4 : b

R : $\{(v_2, v_3)\}$

- compute the "congruence closure" R^c of R
- check whether $(v_1, v_3) \in R^c$

Computing the congruence closure of a DAG

Example

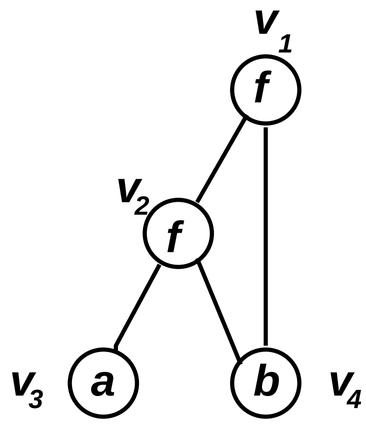
- **DAG structures:**

  - $G = (V, E)$ directed graph

  - Labelling on vertices

    $\lambda(v)$: label of vertex v
    $\delta(v)$: outdegree of vertex v

  - Edges leaving the vertex v are ordered
    ($v[i]$: denotes i-th successor of v)



$\lambda(v_1) = \lambda(v_2) = f$
$\lambda(v_3) = a, \lambda(v_4) = b$

$\delta(v_1) = \delta(v_2) = 2$
$\delta(v_3) = \delta(v_4) = 0$

$v_1[1] = v_2, v_2[2] = v_4$

...

Congruence closure of a DAG/Relation

Given: $G = (V, E)$ DAG + labelling

$R \subseteq V \times V$

The congruence closure of R is the smallest relation R^c on V which is:

- reflexive
- symmetric
- transitive
- congruence:

  If $\lambda(u) = \lambda(v)$ and $\delta(u) = \delta(v)$

  and for all $1 \leq i \leq \delta(u)$: $(u[i], v[i]) \in R^c$

  then $(u, v) \in R^c$.