Decision Procedures in Verification

Decision Procedures (2)

12.12.2013

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Until now:

Decidable subclasses of FOL

The Bernays-Schönfinkel class

(definition; decidability;tractable fragment: Horn clauses)

The Ackermann class

The monadic class

Decision problems/restrictions

Uninterpreted function symbols

Examples

In order to obtain decidability results:

- Restrict the signature
- Enrich axioms
- Look at certain fragments

Decidable theories

- Presburger arithmetic decidable in 3EXPTIME [Presburger'29]
  Signature: $(\{0, 1, +\}, \{\approx, \leq\})$ (no $*$)

  Axioms { (zero), (successor), (induction), (plus zero), (plus successor) }

- Th($\mathbb{Z}_+$) $\mathbb{Z}_+ = (\mathbb{Z}, 0, s, +, \leq)$ the standard interpretation of integers.

Examples

In order to obtain decidability results:

- Restrict the signature
- Enrich axioms
- Look at certain fragments

Decidable theories

- The theory of real numbers (with addition and multiplication) is decidable in 2EXPTIME [Tarski'30]

Examples

In order to obtain decidability results:

- Restrict the signature
- Enrich axioms
- Look at certain fragments

Problems

$\mathcal{T}$: first-order theory in signature Σ; $\mathcal{L}$ class of (closed) Σ-formulae

Given ϕ in $\mathcal{L}$, is it the case that $\mathcal{T} \models \phi$?

Common restrictions on $\mathcal{L}$

	Pred $= \emptyset$	$\{\phi \in \mathcal{L} \mid \mathcal{T} \models \phi\}$
$\mathcal{L}=\{\forall x A(x) \mid A \text{ atomic}\}$	word problem	
$\mathcal{L}=\{\forall x (A_1 \wedge \dots \wedge A_n \rightarrow B) \mid A_i, B \text{ atomic}\}$	uniform word problem	$Th_{\forall \text{Horn}}$
$\mathcal{L}=\{\forall x C(x) \mid C(x) \text{ clause}\}$	clausal validity problem	$Th_{\forall,\text{cl}}$
$\mathcal{L}=\{\forall x \phi(x) \mid \phi(x) \text{ unquantified}\}$	universal validity problem	$Th_{\forall}$
$\mathcal{L}=\{\exists x A_1 \wedge \dots \wedge A_n \mid A_i \text{ atomic}\}$	unification problem	$Th_{\exists}$
$\mathcal{L}=\{\forall x \exists x A_1 \wedge \dots \wedge A_n \mid A_i \text{ atomic}\}$	unification with constants	$Th_{\forall\exists}$

$\mathcal{T}$-validity vs. $\mathcal{T}$-satisfiability

$\mathcal{T}$-validity: Let $\mathcal{T}$ be a first-order theory in signature Σ

Let $\mathcal{L}$ be a class of (closed) Σ-formulae

Given ϕ in $\mathcal{L}$, is it the case that $\mathcal{T} \models \phi$?

Remark: $\mathcal{T} \models \phi$ iff $\mathcal{T} \cup \neg\phi$ unsatisfiable

Every $\mathcal{T}$-validity problem has a dual $\mathcal{T}$-satisfiability problem:

$\mathcal{T}$-satisfiability: Let $\mathcal{T}$ be a first-order theory in signature Σ

Let $\mathcal{L}$ be a class of (closed) Σ-formulae

$\neg\mathcal{L} = \{\neg\phi \mid \phi \in \mathcal{L}\}$

Given ψ in $\neg\mathcal{L}$, is it the case that $\mathcal{T} \cup \psi$ is satisfiable?

$\mathcal{T}$-validity vs. $\mathcal{T}$-satisfiability

Common restrictions on $\mathcal{L}$ / $\neg\mathcal{L}$

$\mathcal{L}$	$\neg\mathcal{L}$
$\{\forall x A(x) \mid A \text{ atomic}\}$	$\{\exists x \neg A(x) \mid A \text{ atomic}\}$
$\{\forall x (A_1 \wedge \ldots \wedge A_n \rightarrow B) \mid A_i, B \text{ atomic}\}$	$\{\exists x (A_1 \wedge \ldots \wedge A_n \wedge \neg B) \mid A_i, B \text{ atomic}\}$
$\{\forall x \bigvee L_i \mid L_i \text{ literals}\}$	$\{\exists x \bigwedge L'_i \mid L'_i \text{ literals}\}$
$\{\forall x \phi(x) \mid \phi(x) \text{ unquantified}\}$	$\{\exists x \phi'(x) \mid \phi'(x) \text{ unquantified}\}$

validity problem for universal formulae

ground satisfiability problem

In what follows we will focus on the problem of checking the satisfiability of conjunctions of ground literals

Theory of Uninterpreted Function Symbols

- Reasoning about equalities is important in automated reasoning
- Applications to program verification
  (approximation: abstract from additional properties)

  Example: Compiler Validation

Solutions

Task:

Check if $UIF \models \forall\overline{x}(s_1(\overline{x})\approx t_1(\overline{x}) \wedge \dots \wedge s_k(\overline{x})\approx t_k(\overline{x}) \rightarrow \bigvee_{j=1}^{m} s'_j(\overline{x})\approx t'_j t(\overline{x}))$

Solution 1:

The following are equivalent:

(1) $(\bigwedge_i s_i \approx t_i) \rightarrow \bigvee_j s'_j \approx t'_j$ is valid

(2) $Eq(\sim) \wedge \mathrm{Con}(f) \wedge (\bigwedge_i s_i \sim t_i) \wedge (\bigwedge_j s'_j \not\sim t'_j)$ is unsatisfiable.

where $Eq(\sim) : \mathrm{Refl}(\sim) \wedge \mathrm{Sim}(\sim) \wedge \mathrm{Trans}(\sim)$

$\mathrm{Con}(f) : \forall x_1, \dots, x_n, y_1, \dots, y_n(\bigwedge x_i \sim y_i \rightarrow f(x_1, \dots, x_n) \sim f(y_1, \dots, y_n))$

Resolution: inferences between transitivity axioms – nontermination

Solutions

Task:

Check if $UIF \models \forall \overline{x}(s_1(\overline{x}) \approx t_1(\overline{x}) \wedge \cdots \wedge s_k(\overline{x}) \approx t_k(\overline{x}) \rightarrow \bigvee_{j=1}^{m} s'_j(\overline{x}) \approx t'_j(\overline{x}))$

Solution 2: Ackermann's reduction.

Flatten the formula (replace, bottom-up, $f(c)$ with a new constant c_f

$\phi \mapsto FLAT(\phi)$

Theorem 3.3.2: The following are equivalent:

(1) $(\bigwedge_i s_i(\overline{c}) \approx t_i(\overline{c})) \wedge \bigwedge_j s'_j(\overline{c}) \not\approx t'_j(\overline{c})$ is satisfiable

(2) $FC \wedge FLAT[(\bigwedge_i s_i(\overline{c}) \approx t_i(\overline{c})) \wedge \bigwedge_j s'_j(\overline{c}) \not\approx t'_j(\overline{c})]$ is satisfiable

where $FC = \{c_1{=}d_1, \dots c_n{=}d_n \rightarrow c_f{=}d_f \mid$ whenever $f(c_1, \dots, c_n)$ was renamed to c_f
$f(d_1, \dots, d_n)$ was renamed to $d_f\}$

Note: The problem is decidable in PTIME

Problem: Naive handling of transitivity/congruence axiom $\mapsto O(n^3)$

Refinements: e.g. rewriting, superposition – not in this lecture

Goal: Give a faster algorithm

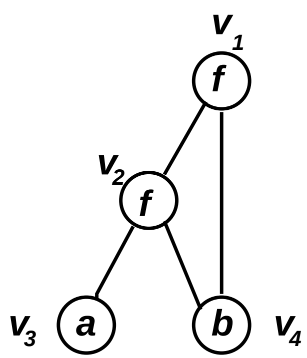
Solutions

Task:
Check if $(s_1(\overline{c}) \approx t_1(\overline{c}) \wedge \cdots \wedge s_k(\overline{c}) \approx t_k(\overline{c}) \wedge \bigwedge_k s'_k(\overline{c}) \not\approx t'_k(\overline{c}))$ unsatisfiable.

Solution 3 [Downey-Sethi, Tarjan'76; Nelson-Oppen'80]

represent the terms occurring in the problem as DAG's

Example: Check whether $f(f(a,b),b) \approx a$ is a consequence of $f(a,b) \approx a$.



v_1 : $f(f(a,b),b)$
v_2 : $f(a,b)$
v_3 : a
v_4 : b

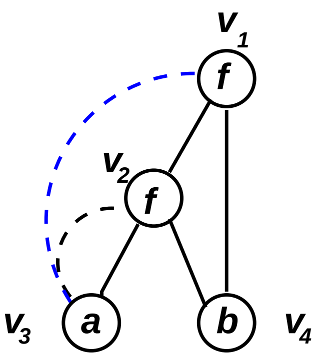
Solutions

Task: Check if $(s_1(\overline{c}) \approx t_1(\overline{c}) \wedge \cdots \wedge s_k(\overline{c}) \approx t_k(\overline{c}) \wedge s(\overline{c}) \not\approx t(\overline{c}))$ unsatisfiable.

Solution 3 [Downey-Sethi, Tarjan'76; Nelson-Oppen'80]

- represent the terms occurring in the problem as DAG's
- represent premise equalities by a relation on the vertices of the DAG

Example: Check whether $f(f(a, b), b) \approx a$ is a consequence of $f(a, b) \approx a$.



v_1 : $f(f(a, b), b)$
v_2 : $f(a, b)$
v_3 : a
v_4 : b
R : $\{(v_2, v_3)\}$

- compute the "congruence closure" R^c of R
- check whether $(v_1, v_3) \in R^c$

Computing the congruence closure of a DAG

Example

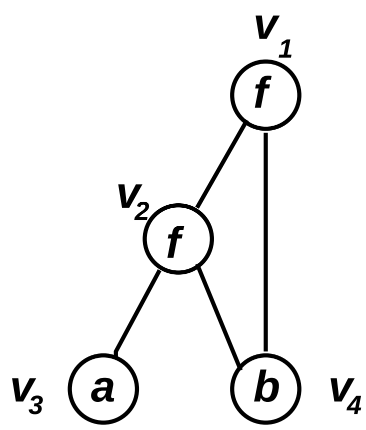
- **DAG structures:**

  - $G = (V, E)$ directed graph

  - Labelling on vertices

    $\lambda(v)$: label of vertex v
    $\delta(v)$: outdegree of vertex v

  - Edges leaving the vertex v are ordered
    ($v[i]$: denotes i-th successor of v)



$\lambda(v_1) = \lambda(v_2) = f$
$\lambda(v_3) = a, \lambda(v_4) = b$

$\delta(v_1) = \delta(v_2) = 2$
$\delta(v_3) = \delta(v_4) = 0$

$v_1[1] = v_2, v_2[2] = v_4$

...

Congruence closure of a DAG/Relation

Given: $G = (V, E)$ DAG + labelling

$R \subseteq V \times V$

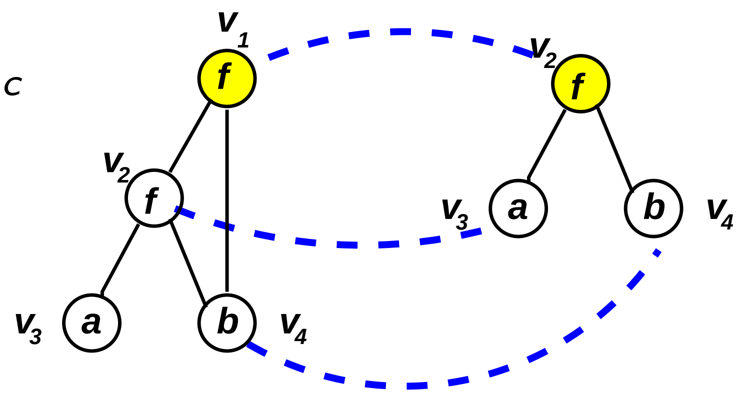
The congruence closure of R is the smallest relation R^c on V which is:

- reflexive
- symmetric
- transitive
- congruence:

  If $\lambda(u) = \lambda(v)$ and $\delta(u) = \delta(v)$

  and for all $1 \leq i \leq \delta(u)$: $(u[i], v[i]) \in R^c$

  then $(u, v) \in R^c$.



Congruence closure of a relation

Recursive definition

$$\frac{(u, v) \in R}{(u, v) \in R^c}$$

$$\frac{}{(v, v) \in R^c} \qquad \frac{(u, v) \in R^c}{(v, u) \in R^c} \qquad \frac{(u, v) \in R^c \quad (v, w) \in R^c}{(u, w) \in R^c}$$

$$\frac{\lambda(u) = \lambda(v) \quad u, v \text{ have } n \text{ successors} \quad \text{and } (u[i], v[i]) \in R^c \text{ for all } 1 \leq i \leq n}{(u, v) \in R^c}$$

- The congruence closure of R is the smallest set closed under these rules

Congruence closure and UIF

Assume that we have an algorithm $\mathbb{A}$ for computing the congruence closure of a graph G and a set R of pairs of vertices

- Use $\mathbb{A}$ for checking whether $\bigwedge_{i=1}^{n} s_i \approx t_i \wedge \bigwedge_{j=1}^{m} s'_j \not\approx t'_j$ is satisfiable.

  (1) Construct graph corresponding to the terms occurring in s_i, t_i, s'_j, t'_j

  Let v_t be the vertex corresponding to term t

  (2) Let $R = \{(v_{s_i}, v_{t_i}) \mid i \in \{1, \dots, n\}\}$

  (3) Compute R^c.

  (4) Output "Sat" if $(v_{s'_j}, v_{t'_j}) \notin R^c$ for all $1 \leq j \leq m$, otherwise "Unsat"

Theorem 3.3.3 (Correctness)

$\bigwedge_{i=1}^{n} s_i \approx t_i \wedge \bigwedge_{j=1}^{m} s'_j \not\approx t'_j$ is satisfiable iff $[v_{s'_j}]_{R^c} \neq [v_{t'_j}]_{R^c}$ for all $1 \leq j \leq m$.

Congruence closure and UIF

Theorem 3.3.3 (Correctness)

$\bigwedge_{i=1}^{n} s_i{\approx}t_i \wedge \bigwedge_{j=1}^{m} s'_j{\not\approx}t'_j$ is satisfiable iff $[v_{s'_j}]_{R^c}{\neq}[v_{t'_j}]_{R^c}$ for all $1{\leq}j{\leq}m$.

Proof ($\Rightarrow$)

Assume $\mathcal{A}$ is a Σ-structure such that $\mathcal{A} \models \bigwedge_{i=1}^{n} s_i \approx t_i \wedge \bigwedge_{j=1}^{m} s'_j \not\approx t'_j$.

We can show that $[v_s]_{R^c} = [v_t]_{R^c}$ implies that $\mathcal{A} \models s = t$ (Exercise).

(We use the fact that if $[v_s]_{R^c} = [v_t]_{R^c}$ then there is a derivation for $(v_s, v_t) \in R^c$ in the calculus defined before; use induction on length of derivation to show that $\mathcal{A} \models s = t$.)

As $\mathcal{A} \models s'_j \not\approx t'_j$, it follows that $[v_{s'_j}]_{R^c}{\neq}[v_{t'_j}]_{R^c}$ for all $1{\leq}j{\leq}m$.

Congruence closure and UIF

Theorem 3.3.3 (Correctness)

$\bigwedge_{i=1}^{n} s_i \approx t_i \wedge \bigwedge_{j=1}^{m} s'_j \not\approx t'_j$ is satisfiable iff $[v_{s'_j}]_{R^c} \neq [v_{t'_j}]_{R^c}$ for all $1 \leq j \leq m$.

Proof($\Leftarrow$) Assume that $[v_{s'_j}]_{R^c} \neq [v_{t'_j}]_{R^c}$ for all $1 \leq j \leq m$. We construct a structure that satisfies $\bigwedge_{i=1}^{n} s_i \approx t_i \wedge \bigwedge_{j=1}^{m} s'_j \not\approx t'_j$

- Universe is quotient of V w.r.t. R^c plus new element 0.
- c constant $\mapsto c_{\mathcal{A}} = [v_c]_{R^c}$.
- $f/n \mapsto f_{\mathcal{A}}([v_1]_{R^c}, \ldots, [v_n]_{R^c}) = \begin{cases} [v_{f(t_1,\ldots,t_n)}]_{R^c} & \text{if } v_{f(t_1,\ldots,t_n)} \in V, \\ & [v_{t_i}]_{R^c} = [v_i]_{R^c} \text{ for } 1 \leq i \leq n \\ 0 & \text{otherwise} \end{cases}$

well-defined because R^c is a congruence.

- It holds that $\mathcal{A} \models s'_j \not\approx t'_j$ and $\mathcal{A} \models s_i \approx t_i$

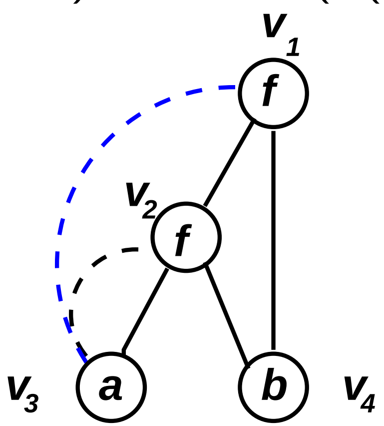
Computing the congruence closure of a DAG

Given: $G = (V, E)$ DAG + labelling
$R \subseteq V \times V$

Task: Compute R^c (the congruence closure of R)

Example:

$f(a, b) \approx a \rightarrow f(f(a, b), b) \approx a$



$R = \{(v_2, v_3)\}$

Idea:

- Start with the identity relation $R^c = Id$
- Successively add new pairs of nodes to R^c;
  close relation under congruence.

Task: Compute R^c

Computing the congruence closure of a DAG

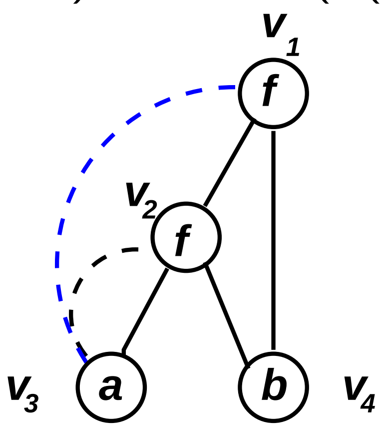
Given: $G = (V, E)$ DAG + labelling

$R \subseteq V \times V$; $(v, v') \in V^2$

Task: Check whether $(v, v') \in R^c$

Example:

$f(a, b) \approx a \rightarrow f(f(a, b), b) \approx a$



$R = \{(v_2, v_3)\}$

Idea:

- Start with the identity relation $R^c = Id$
- Successively add new pairs of nodes to R^c; close relation under congruence.

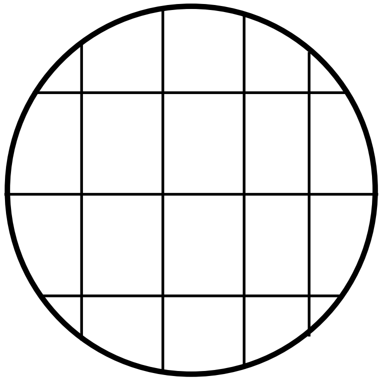
Task: Decide whether $(v_1, v_3) \in R^c$

Computing the congruence closure of a DAG

Given: $G = (V, E)$ DAG + labelling
$R \subseteq V \times V$
Task: Compute R^c (the congruence closure of R)

Idea: Recursively construct relations closed under congruence R_i (approximating R^c) by identifying congruent vertices u, v and computing $R_{i+1} :=$ congruence closure of $R_i \cup \{(u, v)\}$.

Representation:



- Congruence relation $\mapsto$ corresponding partition

Computing the congruence closure of a DAG

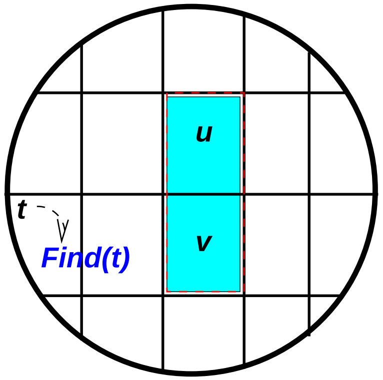
Given: $G = (V, E)$ DAG + labelling

$R \subseteq V \times V$

Task: Compute R^c (the congruence closure of R)

Idea: Recursively construct relations closed under congruence R_i (approximating R^c) by identifying congruent vertices u, v and computing $R_{i+1} :=$ congruence closure of $R_i \cup \{(u, v)\}$.

Representation:



- Congruence relation $\mapsto$ corresponding partition
- Use procedures which operate on the partition:

  FIND(u): unique name of equivalence class of u

  UNION(u, v) combines equivalence classes of u, v

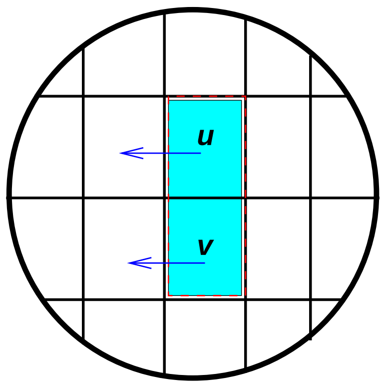
  finds repr. t_u, t_v of equiv.cl. of u, v; sets FIND(u) to t_v

Computing the congruence closure of a DAG

MERGE(u, v)

Input:	$G = (V, E)$ DAG + labelling
	R relation on V closed under congruence
	$u, v \in V$
Output:	the congruence closure of $R \cup \{(u, v)\}$

If FIND(u) $=$ FIND(v) [same canonical representative] **then** Return
If FIND(u) $\neq$ FIND(v) **then** [merge u, v; recursively-predecessors]
 P_u := set of all predecessors of vertices w with FIND(w) $=$ FIND(u)
 P_v := set of all predecessors of vertices w with FIND(w) $=$ FIND(v)
 Call UNION(u, v) [merge congruence classes]
 For all $(x, y) \in P_u \times P_v$ **do**: [merge congruent predecessors]
  if FIND(x) $\neq$ FIND(y) **and** CONGRUENT(x, y) **then** MERGE(x, y)



CONGRUENT(x, y)

if $\lambda(x) \neq \lambda(y)$ **then** Return FALSE
For $1 \leq i \leq \delta(x)$ **if** FIND($x[i]$) $\neq$ FIND($y[i]$) **then** Return FALSE

Return TRUE.

Correctness

Proof:

(1) Returned equivalence relation is not too coarse

If x, y merged then $(x, y) \in (R \cup \{(u, v)\})^c$
(UNION only on initial pair and on congruent pairs)

(2) Returned equivalence relation is not too fine

If x, y vertices s.t. $(x, y) \in (R \cup \{(u, v)\})^c$ then they are merged by the algorithm.
Induction of length of derivation of (x, y) from $(R \cup \{(u, v)\})^c$

(1) $(x, y) \in R$ OK (they are merged)
(2) $(x, y) \notin R$. The only non-trivial case is the following:
$\lambda(x) = \lambda(y), x, y$ have n successors x_i, y_i where
$(x_i, y_i) \in (R \cup \{(u, v)\})^c$ for all $1 \leq i \leq b$.
Induction hypothesis: (x_i, y_i) are merged at some point
(become equal during some call of UNION(a, b), made in some MERGE(a, b))
Successor of x equivalent to a (or b) before this call of UNION; same for y.

$\Rightarrow$ MERGE must merge x and y

Computing the Congruence Closure

Let $G = (V, E)$ graph and $R \subseteq V \times V$

$CC(G, R)$ computes the R^c:

(1) $R_0 := \emptyset$; $i := 1$

(2) while R contains "fresh" elements do:

  pick "fresh" element $(u, v) \in R$

  $R_i :=$ MERGE(u, v) for G and R_{i-1}; $i := i + 1$.

Complexity: $O(n^2)$

Downey-Sethi-Tarjan congruence closure algorithm:

  more sophisticated version of MERGE (complexity $O(n \cdot logn)$)

Reference: G. Nelson and D.C. Oppen. Fast decision procedures based on congruence closure. Journal of the ACM, 27(2):356-364, 1980.

Decision procedure for the QF theory of equality

Signature: Σ (function symbols)

Problem: Test satisfiability of the formula

$$F \quad = \quad s_1 \approx t_1 \wedge \cdots \wedge s_n \approx t_n \quad \wedge \quad s'_1 \not\approx t'_1 \wedge \cdots \wedge s'_m \not\approx t'_m$$

Solution: Let S_F be the set of all subterms occurring in F

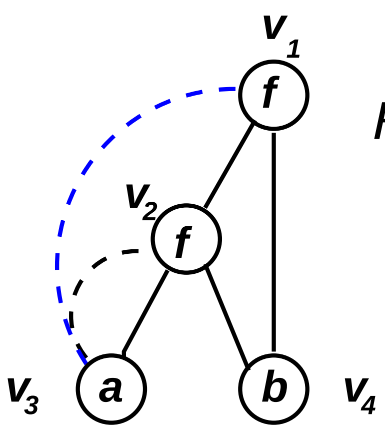
1. Construct the DAG for S_F; $R_0 = Id$
2. [Build R_n the congruence closure of $\{(v(s_1), v(t_1)), \ldots, (v(s_n), v(t_n))\}$]

   For $i \in \{1, \ldots, n\}$ **do** $R_i :=$ MERGE(v_{s_i}, v_{t_i}) w.r.t. R_{i-1}
3. **If** FIND$(v_{s'_j})$ = FIND$(v_{t'_j})$ for some $j \in \{1, \ldots, m\}$ **then** return unsatisfiable
4. **else** [if FIND$(v_{s'_j}) \neq$ FIND$(v_{t'_j})$ for all $j \in \{1, \ldots, m\}$] **then** return satisfiable

Example

$f(a,b) \approx a \rightarrow f(f(a,b),b) \approx a$

Test: unsatisfiability of
$f(a,b) \approx a \wedge f(f(a,b),b) \not\approx a$



$R = \{(v_2, v_3)\}$

Task:

- Compute R^c
- Decide whether $(v_1, v_3) \in R^c$

Solution:

1. Construct DAG in the figure; $R_0 = Id$.
2. Compute $R_1 :=$ MERGE$((v_2, v_3)$

   [Test representatives]

   FIND$(v_2) = v_2 \neq v_3 =$ FIND(v_3)

   $P_{v_2} := \{v_1\}$; $P_{v_3} := \{v_2\}$

   [Merge congruence classes]

   UNION(v_2, v_3): sets FIND(v_2) to v_3.

   [Compute and recursively merge predecessors]

   Test: FIND$(v_1) = v_1 \neq v_3 =$ FIND(v_2)
   CONGR(v_1, v_2)

   MERGE(v_1, v_2): (different representatives)
   calls UNION(v_1, v_2) which
   sets FIND(v_1) to v_3.
3. Test whether FIND$(v_1) =$ FIND(v_3). Yes.
   Return unsatisfiable.

3.4. Decision procedures for numeric domains

- Peano arithmetic
- Theory of real numbers

- Linear arithmetic
  - over $\mathbb{N}/\mathbb{Z}$
  - over $\mathbb{R}/\mathbb{Q}$

Decision procedures

- Light-weight fragments of linear arithmetic: Difference logic
- Full fragment ($LI(\mathbb{R})$ or $LI(\mathbb{Q})$)

Peano arithmetic

Peano axioms:

$\forall x\, \neg(x + 1 \approx 0)$	(zero)
$\forall x \forall y\, (x + 1 \approx y + 1 \to x \approx y)$	(successor)
$F[0] \wedge (\forall x\, (F[x] \to F[x + 1]) \to \forall x F[x])$	(induction)
$\forall x\, (x + 0 \approx x)$	(plus zero)
$\forall x, y\, (x + (y + 1) \approx (x + y) + 1)$	(plus successor)
$\forall x, y\, (x * 0 \approx 0)$	(times 0)
$\forall x, y\, (x * (y + 1) \approx x * y + x)$	(times successor)

$3 * y + 5 > 2 * y$ expressed as $\exists z (z \neq 0 \wedge 3 * y + 5 \approx 2 * y + z)$

Intended interpretation: $(\mathbb{N}, \{0, 1, +, *\}, \{<\})$ (also with $\approx$)

(does not capture true arithmetic by Goedel's incompleteness theorem)

Undecidable

Theory of integers

• $\mathsf{Th}((\mathbb{Z}, \{0, 1, +, *\}, \{<\}))$

Undecidable

Theory of real numbers

Theory of real closed fields (real closed fields: fields with same properties as real numbers)

Axioms:

- the ordered field axioms;
- axiom asserting that every positive number has a square root; and
- an axiom scheme asserting that all polynomials of odd order have at least one real root.

Tarski (1951) proved that the theory of real closed fields, including the binary predicate symbols "$=$", "$\neq$", and "$<$", and the operations of addition and multiplication, admits elimination of quantifiers, which implies that it is a complete and decidable theory.

Linear arithmetic

Syntax

- Signature $\Sigma = (\{0/0, s/1, +/2\}, \{< /2\})$
- Terms, atomic formulae – as usual

Example: $3 * x_1 + 2 * x_2 \leq 5 * x_3$ abbreviation for

$$(x_1 + x_1 + x_1) + (x_2 + x_2) \leq (x_3 + x_3 + x_3 + x_3 + x_3)$$

Linear arithmetic

There are several ways to define linear arithmetic.

We need at least the following signature: $\Sigma = (\{0/0, 1/0, +/2\}, \{< /2\})$ and the predefined binary predicate $\approx$.

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Linear arithmetic over $\mathbb{N}/\mathbb{Z}$

Th($\mathbb{Z}_+$) $\quad \mathbb{Z}_+ = (\mathbb{Z}, 0, s, +, <)$ the standard interpretation of integers.

Axiomatization

Linear arithmetic over $\mathbb{Q}/\mathbb{R}$

Th($\mathbb{R}$) $\quad \mathbb{R} = (\mathbb{R}, \{0, 1, +\}, \{<\})$ the standard interpretation of reals;

Th($\mathbb{Q}$) $\quad \mathbb{Q} = (\mathbb{Q}, \{0, 1, +\}, \{<\})$ the standard interpretation of rationals.

Axiomatization

Outline

We first present an efficient method for checking the satisfiability of formulae in a very simple fragment of linear arithmetic.

We will then give more details about possibilities of checking the satisfiability of arbitrary formulae in linear arithmetic.

Simple fragments of linear arithmetic

- Difference logic

  check satisfiability of conjunctions of constraints of the form

$$x - y \leq c$$

- UTVPI (unit, two variables per identity)

  check satisfiability of conjunctions of constraints of the form

$$ax + by \leq c, \text{ where } a, b \in \{-1, 0, 1\}$$

Application: Program Verification

```
i := 1,  n < m
while i < n
do
   i := i + 1
    [** part of a program in which  i  is used as an index in an array
        which was declared to be of size s > m (and i is not changed)
     **]
   ....
od
```

Task: $i \leq s$ always during the execution of this program.

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Solution: Show that this is true at the beginning and remains true after every update of i.

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```

Task: $i \leq s$ always during the execution of this program.

Solution: Show that $i \leq s$ is an invariant of the program:

1) It holds at the first line: $i = 1 \rightarrow i \leq s$

2) It is preserved under the updates in the while loop:
$\forall n, m, s, i, i' \ \ (n < m \wedge 1 < m < s \wedge i \leq n \wedge i \leq s \wedge i' \approx i + 1 \rightarrow i' \leq s)$

Positive difference logic

Syntax

The syntax of formulae in positive difference logic is defined as follows:

- Atomic formulae (also called difference constraints) are of the form:

  $x - y \leq c$

  where x, y are variables and c is a numerical constant.

- The set of formulae is:

$$
\begin{array}{lllr}
F, G, H & ::= & \mathrm{A} & \text{(atomic formula)} \\
 & | & (F \wedge G) & \text{(conjunction)}
\end{array}
$$

Semantics:

Versions of difference logic exist, where the standard interpretation is $\mathbb{Q}$ or resp. $\mathbb{Z}$.

Positive difference logic

A decision procedure for positive difference logic ($\leq$ only)

Let S be a set (i.e. conjunction) of atoms in (positive) difference logic. $G(S) = (V, E, w)$, the inequality graph of S, is a weighted graph with:

- $V = X(S)$, the set of variables occurring in S
- $e = (x, y) \in E$ with $w(e) = c$ iff $x - y \leq c \in S$

Theorem 3.4.1.
Let S be a conjunction of difference constraints, and $G(S)$ the inequality graph of S. Then S is satisfiable iff there is no negative cycle in $G(S)$.

Searching for negative cycles in a graph can be done with the Bellman-Ford algorithm for finding the single-source shortest paths in a directed weighted graph in time $O(|V| \cdot |E|)$. (Side-effect of the algorithm exploited - if there exists a negative cycle in the graph then the algorithm finds it and aborts.)

Positive difference logic

Theorem 3.4.1.
Let S be a conjunction of difference constraints, and $G(S)$ the inequality graph of S. Then S is satisfiable iff there is no negative cycle in $G(S)$.

Proof: ($\Rightarrow$) Assume S satisfiable. Let $\beta : X \rightarrow \mathbb{Z}$ satisfying assignment.
Let $v_1 \stackrel{c_{12}}{\rightarrow} v_2 \stackrel{c_{23}}{\rightarrow} \cdots \stackrel{c_{n-1,n}}{\rightarrow} v_n \stackrel{c_{n1}}{\rightarrow} v_1$ be a cycle in $G(S)$.

$$
\begin{array}{rccl}
\text{Then:} & \beta(v_1) - \beta(v_2) & \leq & c_{12} \\
& \beta(v_2) - \beta(v_3) & \leq & c_{23} \\
& \dots & & \\
g & \beta(v_n) - \beta(v_1) & \leq & c_{n1} \\
\hline
0 = & \beta(v_1) - \beta(v_1) & \leq & \sum_{i=1}^{n-1} c_{i,i+1} + c_{n1}
\end{array}
$$

Thus, for satisfiability it is necessary that all cycles are positive.

Positive difference logic

Theorem 3.4.1.
Let S be a conjunction of difference constraints, and $G(S)$ the inequality graph of S. Then S is satisfiable iff there is no negative cycle in $G(S)$.

Proof: $(\Leftarrow)$ Assume that there is no negative cycle.

Add a new vertex s and an 0-weighted edge from every vertex in V to s. (This does not introduce negative cycles.)

Let δ_{uv} denote the minimal weight of the paths from u to v.

- $\delta_{uv} = \infty$ if there is no path from u to v.
- well-defined since there are no negative cycles

Define $\beta : V \to \mathbb{Z}$ by $\beta(v) = \delta_{vs}$. Claim: β satisfying assignment for S.

Let $x - y \leq c \in S$. Consider the shortest paths from x to s and from y to s. By the triangle inequality, $\delta_{xs} \leq c + \delta_{ys}$, i.e. $\beta(x) - \beta(y) \leq c$.