Decision Procedures for Verification

Decision Procedures (2)

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Exam

Thursday, 12.03.2015, 13:00-15:00

Until now:

Decision Procedures

- Uninterpreted functions

  congruence closure

3.3. Theory of Uninterpreted Function Symbols

Why?

- Reasoning about equalities is important in automated reasoning
- Applications to program verification
  (approximation: abstract from additional properties)

Application: Compiler Validation

Example: prove equivalence of source and target program

```
1:  y := 1
2:  if z = x*x*x
3:    then y := x*x + y
4:  endif
```

```
1: y := 1
2: R1 := x*x
3: R2 := R1*x
4: jmpNE(z,R2,6)
5: y := R1+1
```

To prove: (indexes refer to values at line numbers)

$$
\begin{aligned}
& y_1 \approx 1 \wedge [(z_0 \approx x_0 * x_0 * x_0 \wedge y_3 \approx x_0 * x_0 + y_1) \vee (z_0 \not\approx x_0 * x_0 * x_0 \wedge y_3 \approx y_1)] \wedge \\
& y'_1 \approx 1 \wedge R1_2 \approx x'_0 * x'_0 \wedge R2_3 \approx R1_2 * x'_0 \wedge \\
& \qquad \wedge [(z'_0 \approx R2_3 \wedge y'_5 \approx R1_2 + 1) \vee (z'_0 \neq R2_3 \wedge y'_5 \approx y'_1)] \wedge \\
& x_0 \approx x'_0 \wedge y_0 \approx y'_0 \wedge z_0 \approx z'_0 \implies x_0 \approx x'_0 \wedge y_3 \approx y'_5 \wedge z_0 \approx z'_0
\end{aligned}
$$

Possibilities for checking it

(1) **Abstraction.**
Consider $*$ to be a "free" function symbol (forget its properties). Test it property can be proved in this approximation. If so, then we know that implication holds also under the normal interpretation of $*$.

(2) **Reasoning about formulae in fragments of arithmetic.**

Uninterpreted function symbols

Let $\Sigma = (\Omega, \Pi)$ be arbitrary

Let $\mathcal{M} = \Sigma$-alg be the class of all Σ-structures

The theory of uninterpreted function symbols is Th(Σ-alg) the family of all first-order formulae which are true in all Σ-algebras.

in general undecidable

Decidable fragment:

e.g. the class $\mathrm{Th}_\forall(\Sigma\text{-alg})$ of all universal formulae which are true in all Σ-algebras.

Assume $\Pi = \emptyset$ (and $\approx$ is the only predicate)

In this case we denote the theory of uninterpreted function symbols by $UIF(\Sigma)$ (or UIF when the signature is clear from the context).

This theory is sometimes called the theory of free functions and denoted Free(Σ)

Uninterpreted function symbols

Theorem 3.3.1

The following are equivalent:

(1) testing validity of universal formulae w.r.t. UIF is decidable

(2) testing validity of (universally quantified) clauses w.r.t. UIF is decidable

(3) testing satisfiability of conjunctions of literals w.r.t. UIF is decidable

Task:
Check if $UIF \models \forall \overline{x}(s_1(\overline{x}) \approx t_1(\overline{x}) \wedge \cdots \wedge s_k(\overline{x}) \approx t_k(\overline{x}) \rightarrow \bigvee_{j=1}^{m} s'_j(\overline{x}) \approx t'_j t(\overline{x}))$

Solutions

Solution 1. The following are equivalent:

(1) $(\bigwedge_i s_i \approx t_i) \rightarrow \bigvee_j s'_j \approx t'_j$ is valid

(2) $Eq(\sim) \wedge \mathrm{Con}(f) \wedge (\bigwedge_i s_i{\sim}t_i) \wedge (\bigwedge_j s'_j \not\sim t'_j)$ is unsatisfiable.

where $Eq(\sim)$: $\mathrm{Refl}(\sim) \wedge \mathrm{Sim}(\sim) \wedge \mathrm{Trans}(\sim)$

$\mathrm{Con}(f) : \forall x_1, \dots, x_n, y_1, \dots, y_n (\bigwedge x_i{\sim}y_i {\rightarrow} f(x_1, \dots, x_n) \sim f(y_1, \dots, y_n))$

Disadvantage: Resolution inferences between transitivity axioms – nontermination

Solution 2. Ackermann's reduction: Flatten the formula (replace, bottom-up, $f(c)$ with a new constant c_f) $\quad \phi \mapsto FLAT(\phi)$

Theorem 3.3.2: The following are equivalent:

(1) $(\bigwedge_i s_i(\overline{c}) \approx t_i(\overline{c})) \wedge \bigwedge_j s'_j(\overline{c}) \not\approx t'_j(\overline{c})$ is satisfiable

(2) $FC \wedge FLAT[(\bigwedge_i s_i(\overline{c}) \approx t_i(\overline{c})) \wedge \bigwedge_j s'_j(\overline{c}) \not\approx t'_j(\overline{c})]$ is satisfiable

where $FC = \{c_1{\approx}d_1, \dots c_n{\approx}d_n \rightarrow c_f{\approx}d_f \mid$ if $f(c_1, \dots, c_n)$ was renamed to c_f
$f(d_1, \dots, d_n)$ was renamed to $d_f\}$

Note: The problem is decidable in PTIME

Problem: Handling of transitivity/congruence axiom $\mapsto O(n^3)$

Example

The following are equivalent:

(1) $C := f(a,b) \approx a \wedge f(f(a,b),b) \not\approx a$ is satisfiable

(2) $FC \wedge FLAT[C]$ is satisfiable, where:

$FLAT[f(a,b) \approx a \wedge f(f(a,b),b) \not\approx a]$ is computed by introducing new constants renaming terms starting with f and then replacing in C the terms with the constants:

- $FLAT[\underbrace{f(a,b)}_{a_1} \approx a \wedge \underbrace{f(\underbrace{f(a,b)}_{a_1},b)}_{a_2} \not\approx a] := a_1 \approx a \wedge a_2 \not\approx a$

$$f(a,b) = a_1$$
$$f(a_1,b) = a_2$$

- $FC := (a \approx a_1 \rightarrow a_1 \approx a_2)$

Thus, the following are equivalent:

(1) $C := f(a,b) \approx a \wedge f(f(a,b),b) \not\approx a$ is satisfiable

(2) $\underbrace{(a \approx a_1 \rightarrow a_1 \approx a_2)}_{FC} \wedge \underbrace{a_1 \approx a \wedge a_2 \not\approx a}_{FLAT[C]}$ is satisfiable

Problems: Handling $\approx$; Redundancy in representation

Goal: Better algorithm

Solution 3

Task:

Check if $UIF \models \forall\overline{x}(s_1(\overline{x})\approx t_1(\overline{x}) \wedge \cdots \wedge s_k(\overline{x})\approx t_k(\overline{x}) \rightarrow \bigvee_{j=1}^{m} s'_j(\overline{x})\approx t'_j(\overline{x}))$

i.e. if $(s_1(\overline{c})\approx t_1(\overline{c}) \wedge \cdots \wedge s_k(\overline{c})\approx t_k(\overline{c}) \wedge \bigwedge_j s'_j(\overline{c})\not\approx t'_j(\overline{c}))$ unsatisfiable.

Solution 3

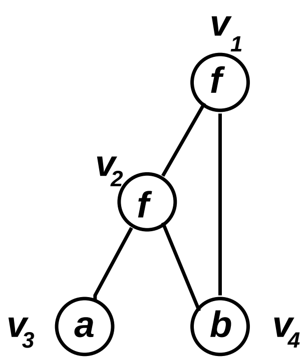
Task:

Check if $(s_1(\overline{c}) \approx t_1(\overline{c}) \wedge \cdots \wedge s_k(\overline{c}) \approx t_k(\overline{c}) \wedge \bigwedge_k s'_k(\overline{c}) \not\approx t'_k(\overline{c}))$ unsatisfiable.

Solution 3 [Downey-Sethi, Tarjan'76; Nelson-Oppen'80]

represent the terms occurring in the problem as DAG's

Example: Check whether $f(f(a, b), b) \approx a$ is a consequence of $f(a, b) \approx a$.



$$
\begin{array}{lll}
v_1 & : & f(f(a, b), b) \\
v_2 & : & f(a, b) \\
v_3 & : & a \\
v_4 & : & b
\end{array}
$$

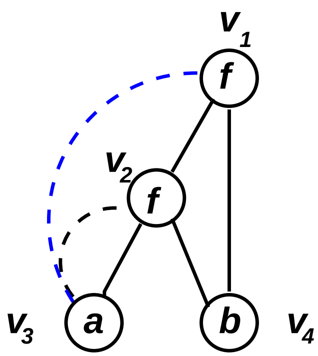
Solution 3

Task: Check if $(s_1(\overline{c}) \approx t_1(\overline{c}) \wedge \cdots \wedge s_k(\overline{c}) \approx t_k(\overline{c}) \wedge s(\overline{c}) \not\approx t(\overline{c}))$ unsatisfiable.

Solution 3 [Downey-Sethi, Tarjan'76; Nelson-Oppen'80]

- represent the terms occurring in the problem as DAG's
- represent premise equalities by a relation on the vertices of the DAG

Example: Check whether $f(f(a, b), b) \approx a$ is a consequence of $f(a, b) \approx a$.



$$
\begin{array}{ll}
v_1 : & f(f(a, b), b) \\
v_2 : & f(a, b) \\
v_3 : & a \\
v_4 : & b \\
R : & \{(v_2, v_3)\}
\end{array}
$$

- compute the "congruence closure" R^c of R
- check whether $(v_1, v_3) \in R^c$

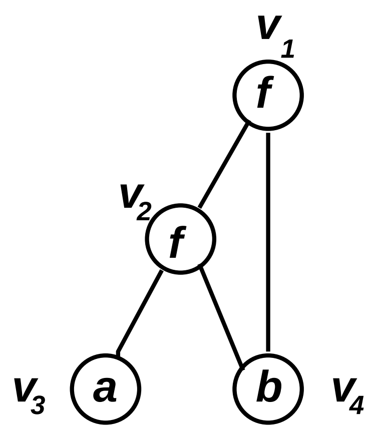
Computing the congruence closure of a DAG

Example

- **DAG structures:**
  - $G = (V, E)$ directed graph
  - Labelling on vertices

    $\lambda(v)$: label of vertex v

    $\delta(v)$: outdegree of vertex v
  - Edges leaving the vertex v are ordered
    ($v[i]$: denotes i-th successor of v)



$\lambda(v_1) = \lambda(v_2) = f$
$\lambda(v_3) = a, \lambda(v_4) = b$

$\delta(v_1) = \delta(v_2) = 2$
$\delta(v_3) = \delta(v_4) = 0$

$v_1[1] = v_2, v_2[2] = v_4$

...

Congruence closure of a DAG/Relation

Given: $G = (V, E)$ DAG + labelling

$R \subseteq V \times V$

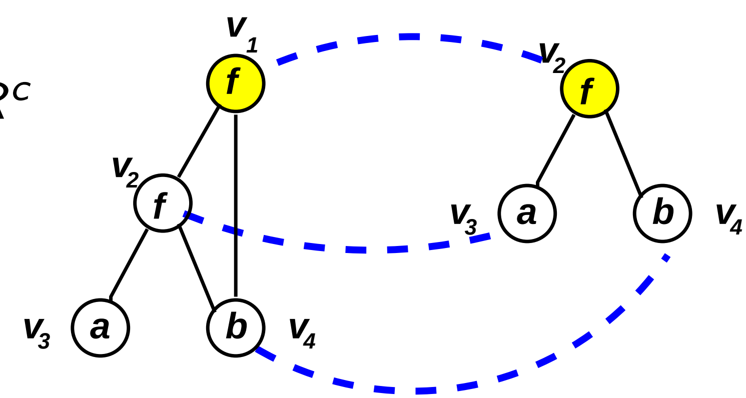
The congruence closure of R is the smallest relation R^c on V which is:

- reflexive
- symmetric
- transitive
- congruence:

  If $\lambda(u) = \lambda(v)$ and $\delta(u) = \delta(v)$

  and for all $1 \leq i \leq \delta(u)$: $(u[i], v[i]) \in R^c$

  then $(u, v) \in R^c$.



Congruence closure of a relation

Recursive definition

$$\frac{(u, v) \in R}{(u, v) \in R^c}$$

$$\frac{}{(v, v) \in R^c} \qquad \frac{(u, v) \in R^c}{(v, u) \in R^c} \qquad \frac{(u, v) \in R^c \quad (v, w) \in R^c}{(u, w) \in R^c}$$

$$\frac{\lambda(u) = \lambda(v) \quad u, v \text{ have } n \text{ successors} \quad \text{and } (u[i], v[i]) \in R^c \text{ for all } 1 \leq i \leq n}{(u, v) \in R^c}$$

- The congruence closure of R is the smallest set closed under these rules

Congruence closure and UIF

Assume that we have an algorithm $\mathbb{A}$ for computing the congruence closure of a graph G and a set R of pairs of vertices

- Use $\mathbb{A}$ for checking whether $\bigwedge_{i=1}^{n} s_i \approx t_i \wedge \bigwedge_{j=1}^{m} s'_j \not\approx t'_j$ is satisfiable.

  (1) Construct graph corresponding to the terms occurring in s_i, t_i, s'_j, t'_j

  Let v_t be the vertex corresponding to term t

  (2) Let $R = \{(v_{s_i}, v_{t_i}) \mid i \in \{1, \dots, n\}\}$

  (3) Compute R^c.

  (4) Output "Sat" if $(v_{s'_j}, v_{t'_j}) \notin R^c$ for all $1 \leq j \leq m$, otherwise "Unsat"

Theorem 3.3.3 (Correctness)

$\bigwedge_{i=1}^{n} s_i \approx t_i \wedge \bigwedge_{j=1}^{m} s'_j \not\approx t'_j$ is satisfiable iff $[v_{s'_j}]_{R^c} \neq [v_{t'_j}]_{R^c}$ for all $1 \leq j \leq m$.

Congruence closure and UIF

Theorem 3.3.3 (Correctness)

$\bigwedge_{i=1}^{n} s_i \approx t_i \wedge \bigwedge_{j=1}^{m} s'_j \not\approx t'_j$ is satisfiable iff $[v_{s'_j}]_{R^c} \neq [v_{t'_j}]_{R^c}$ for all $1 \leq j \leq m$.

Proof ($\Rightarrow$)

Assume $\mathcal{A}$ is a Σ-structure such that $\mathcal{A} \models \bigwedge_{i=1}^{n} s_i \approx t_i \wedge \bigwedge_{j=1}^{m} s'_j \not\approx t'_j$.

We can show that $[v_s]_{R^c} = [v_t]_{R^c}$ implies that $\mathcal{A} \models s = t$ (Exercise).

(We use the fact that if $[v_s]_{R^c} = [v_t]_{R^c}$ then there is a derivation for $(v_s, v_t) \in R^c$ in the calculus defined before; use induction on length of derivation to show that $\mathcal{A} \models s = t$.)

As $\mathcal{A} \models s'_j \not\approx t'_j$, it follows that $[v_{s'_j}]_{R^c} \neq [v_{t'_j}]_{R^c}$ for all $1 \leq j \leq m$.

Congruence closure and UIF

Theorem 3.3.3 (Correctness)

$\bigwedge_{i=1}^{n} s_i \approx t_i \wedge \bigwedge_{j=1}^{m} s'_j \not\approx t'_j$ is satisfiable iff $[v_{s'_j}]_{R^c} \neq [v_{t'_j}]_{R^c}$ for all $1 \leq j \leq m$.

Proof($\Leftarrow$) Assume that $[v_{s'_j}]_{R^c} \neq [v_{t'_j}]_{R^c}$ for all $1 \leq j \leq m$. We construct a structure that satisfies $\bigwedge_{i=1}^{n} s_i \approx t_i \wedge \bigwedge_{j=1}^{m} s'_j \not\approx t'_j$

- Universe is quotient of V w.r.t. R^c plus new element 0.
- c constant $\mapsto c_{\mathcal{A}} = [v_c]_{R^c}$.
- $f/n \mapsto f_{\mathcal{A}}([v_1]_{R^c}, \ldots, [v_n]_{R^c}) = \begin{cases} [v_{f(t_1,\ldots,t_n)}]_{R^c} & \text{if } v_{f(t_1,\ldots,t_n)} \in V, \\ & [v_{t_i}]_{R^c} = [v_i]_{R^c} \text{ for } 1 \leq i \leq n \\ 0 & \text{otherwise} \end{cases}$

well-defined because R^c is a congruence.

- It holds that $\mathcal{A} \models s'_j \not\approx t'_j$ and $\mathcal{A} \models s_i \approx t_i$

Computing the congruence closure of a DAG

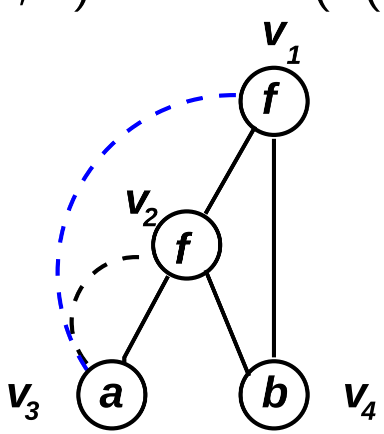
Given: $G = (V, E)$ DAG + labelling

$R \subseteq V \times V$

Task: Compute R^c (the congruence closure of R)

Example:

$f(a, b) \approx a \rightarrow f(f(a, b), b) \approx a$



$R = \{(v_2, v_3)\}$

Idea:

- Start with the identity relation $R^c = Id$
- Successively add new pairs of nodes to R^c; close relation under congruence.

Task: Compute R^c

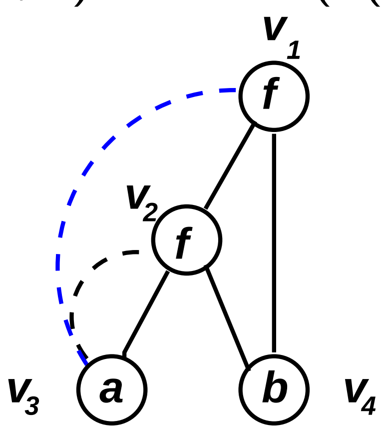
Computing the congruence closure of a DAG

Given: $G = (V, E)$ DAG + labelling
$R \subseteq V \times V$; $(v, v') \in V^2$

Task: Check whether $(v, v') \in R^c$

Example:

$f(a, b) \approx a \rightarrow f(f(a, b), b) \approx a$



$R = \{(v_2, v_3)\}$

Idea:

- Start with the identity relation $R^c = Id$
- Successively add new pairs of nodes to R^c; close relation under congruence.

Task: Decide whether $(v_1, v_3) \in R^c$

Computing the congruence closure of a DAG

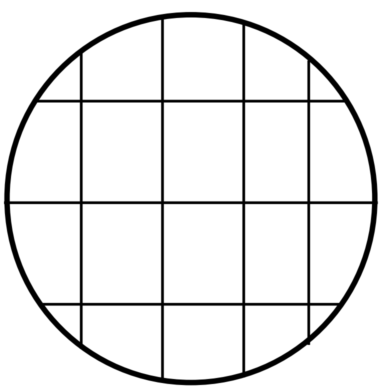
Given: $G = (V, E)$ DAG + labelling

$R \subseteq V \times V$

Task: Compute R^c (the congruence closure of R)

Idea: Recursively construct relations closed under congruence R_i (approximating R^c) by identifying congruent vertices u, v and computing $R_{i+1} :=$ congruence closure of $R_i \cup \{(u, v)\}$.

Representation:



- Congruence relation $\mapsto$ corresponding partition

Computing the congruence closure of a DAG

Given: $G = (V, E)$ DAG + labelling

$R \subseteq V \times V$

Task: Compute R^c (the congruence closure of R)

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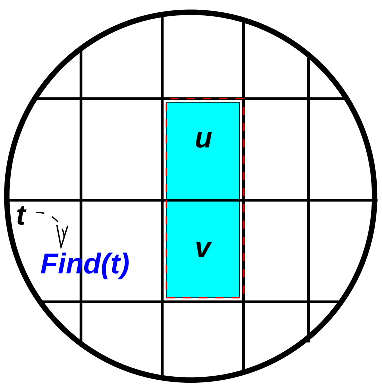


- Congruence relation $\mapsto$ corresponding partition
- Use procedures which operate on the partition:

  FIND(u): unique name of equivalence class of u

  UNION(u, v) combines equivalence classes of u, v

  finds repr. t_u, t_v of equiv.cl. of u, v; sets FIND(u) to

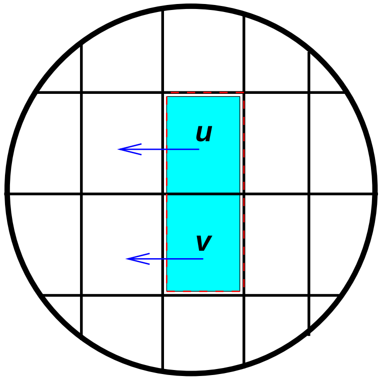
Computing the congruence closure of a DAG

MERGE(u, v)

g

Input:	$G = (V, E)$ DAG + labelling
	R relation on V closed under congruence
	$u, v \in V$
Output:	the congruence closure of $R \cup \{(u, v)\}$

If FIND(u) = FIND(v) [same canonical representative] **then** Return
If FIND(u) $\neq$ FIND(v) **then** [merge u, v; recursively-predecessors]
P_u := set of all predecessors of vertices w with FIND(w) = FIND(u)
P_v := set of all predecessors of vertices w with FIND(w) = FIND(v)
Call UNION(u, v) [merge congruence classes]
For all $(x, y) \in P_u \times P_v$ **do**: [merge congruent predecessors]
if FIND(x) $\neq$ FIND(y) **and** CONGRUENT(x, y) **then** MERGE(x, y)



CONGRUENT(x, y)

if $\lambda(x) \neq \lambda(y)$ **then** Return FALSE
For $1 \leq i \leq \delta(x)$ **if** FIND($x[i]$) $\neq$ FIND($y[i]$) **then** Return FALSE

Return TRUE.

Correctness

Proof:

(1) Returned equivalence relation is not too coarse

If x, y merged then $(x, y) \in (R \cup \{(u, v)\})^c$
(UNION only on initial pair and on congruent pairs)

(2) Returned equivalence relation is not too fine

If x, y vertices s.t. $(x, y) \in (R \cup \{(u, v)\})^c$ then they are merged by the algorithm.
Induction of length of derivation of (x, y) from $(R \cup \{(u, v)\})^c$

(1) $(x, y) \in R$ OK (they are merged)
(2) $(x, y) \notin R$. The only non-trivial case is the following:
$\lambda(x) = \lambda(y), x, y$ have n successors x_i, y_i where
$(x_i, y_i) \in (R \cup \{(u, v)\})^c$ for all $1 \leq i \leq b$.
Induction hypothesis: (x_i, y_i) are merged at some point

(become equal during some call of UNION(a, b), made in some MERGE(a, b))
Successor of x equivalent to a (or b) before this call of UNION; same for y.

$\Rightarrow$ MERGE must merge x and y

Computing the Congruence Closure

Let $G = (V, E)$ graph and $R \subseteq V \times V$

$CC(G, R)$ computes the R^c:

(1) $R_0 := \emptyset$; $i := 1$

(2) while R contains "fresh" elements do:

pick "fresh" element $(u, v) \in R$

$R_i :=$ MERGE(u, v) for G and R_{i-1}; $i := i + 1$.

Complexity: $O(n^2)$
Downey-Sethi-Tarjan congruence closure algorithm:
more sophisticated version of MERGE (complexity $O(n \cdot logn)$)

Reference: G. Nelson and D.C. Oppen. Fast decision procedures based on congruence closure. Journal of the ACM, 27(2):356-364, 1980.

Decision procedure for the QF theory of equality

Signature: Σ (function symbols)

Problem: Test satisfiability of the formula

$$F \quad = \quad s_1 \approx t_1 \wedge \cdots \wedge s_n \approx t_n \quad \wedge \quad s'_1 \not\approx t'_1 \wedge \cdots \wedge s'_m \not\approx t'_m$$

Solution: Let S_F be the set of all subterms occurring in F

1. Construct the DAG for S_F; $R_0 = Id$
2. [Build R_n the congruence closure of $\{(v(s_1), v(t_1)), \dots, (v(s_n), v(t_n))\}$]

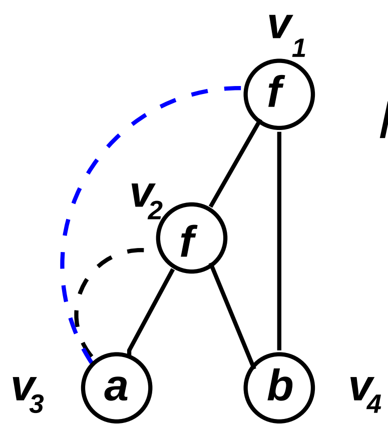
   For $i \in \{1, \dots, n\}$ **do** $R_i :=$ MERGE(v_{s_i}, v_{t_i}) w.r.t. R_{i-1}
3. **If** FIND($v_{s'_j}$) = FIND($v_{t'_j}$) for some $j \in \{1, \dots, m\}$ **then** return unsatisfiable
4. **else** [if FIND($v_{s'_j}$) $\neq$ FIND($v_{t'_j}$) for all $j \in \{1, \dots, m\}$] **then** return satisfiable

Example

$f(a,b) \approx a \rightarrow f(f(a,b),b) \approx a$

Test: unsatisfiability of

$f(a,b) \approx a \wedge f(f(a,b),b) \not\approx a$



$R = \{(v_2, v_3)\}$

Task:

- Compute R^c
- Decide whether $(v_1, v_3) \in R^c$

Solution:

1. Construct DAG in the figure; $R_0 = Id$.
2. Compute $R_1 := \text{MERGE}((v_2, v_3)$

    [Test representatives]

    $\text{FIND}(v_2) = v_2 \neq v_3 = \text{FIND}(v_3)$

    $P_{v_2} := \{v_1\}; P_{v_3} := \{v_2\}$

    [Merge congruence classes]

    $\text{UNION}(v_2, v_3)$: sets $\text{FIND}(v_2)$ to v_3.

    [Compute and recursively merge predecessors]

    Test: $\text{FIND}(v_1) = v_1 \neq v_3 = \text{FIND}(v_2)$

    $\text{CONGR}(v_1, v_2)$

    $\text{MERGE}(v_1, v_2)$: (different representatives)

    calls $\text{UNION}(v_1, v_2)$ which

    sets $\text{FIND}(v_1)$ to v_3.

3. Test whether $\text{FIND}(v_1) = \text{FIND}(v_3)$. Yes.

    Return unsatisfiable.