Universität Koblenz-Landau

FB 4 Informatik

Prof. Dr. Viorica Sofronie-Stokkermans **November 14 2016**

Exercises for "Decision Procedures for Verification"
Exercise sheet 4

Exercise 4.1: (*2 P*)
Assume $P \succ Q \succ R$. Let N be the following set of clauses:

$$\begin{array}{ll} (1) & \neg R \vee P \\ (2) & \neg Q \vee \neg P \\ (3) & Q \\ (4) & R \vee P \end{array}$$

Let S be the selection function which selects $\neg R$ in clause (1) and $\neg Q$ in clause (2).

Use the ordered resolution calculus with selection $\mathsf{Res}_S^{\succ}$ described in the lecture for checking the satisfiability of the set N of clauses.

Exercise 4.2: (*2 P*)
A propositional Horn clause is a clause which has at most one positive literal.
(*Example:* $\neg P \vee Q \vee \neg R$, $\neg P \vee \neg R$ and Q are Horn clauses,
whereas $\neg P \vee Q \vee R$ and $Q \vee R$ are not Horn clauses.)
Prove: Every set H of clauses with the following properties:

(i) H consists only of Horn clauses;

(ii) Every clause in H contains at least one negative literal;

is satisfiable.

Exercise 4.3: (*5 P*)
Let H be a set of propositional Horn clauses. The size of H is the number of all literals which occur in H. Prove that the resolution calculus $\mathsf{Res}_S^{\succ}$ (or the marking algorithm discussed in the lecture "Logik für Informatiker") can check the satisfiability of H in time polynomial in the size of H.

Supplementary question: Can you give an algorithm for check the satisfiability of H in time linear in the size of H?

Exercise 4.4: (*2 P*)
Let $\Sigma = (\Omega, \Pi)$ be a signature, where $\Omega = \{f/3,\ g/1,\ a/0,\ b/0\}$ and $\Pi = \{p/2\}$; let X be the set of variables $\{x, y, z\}$. Which of the following expressions are terms over Σ and X, which are atoms/literals/clauses/formulae (in first-order logic with equality), which are neither?

(a) $\neg p(g(a), f(x, y, g(a)))$
(b) $f(x, x, x) \approx x$
(c) $p(f(x, x, a), x) \vee p(a, b)$
(d) $p(\neg g(x), g(y))$
(e) $\neg p(f(x, y, y))$
(f) $\neg p(f(x, y), y) \vee p(x, y)$
(g) $p(a, b) \wedge p(x, y) \wedge y$
(h) $\exists y(\neg p(f(y, y, y), y))$
(i) $\forall x \forall y(f(p(x, y), x, x) \approx g(x))$

You will be able to solve this exercise only after the terminology is introduced in the lecture on Monday, 21.11.2016.

Exercise 4.5: *(2 P)*
Let $\Sigma = (S, \Omega, \Pi)$ be a many-sorted signature, where $S = \{\mathsf{int}, \mathsf{list}\}$, $\Omega = \{\mathsf{cons}, \mathsf{car}, \mathsf{cdr}, \mathsf{nil}, b\}$ and $\Pi = \{p\}$ with the following arities:

$a(\mathsf{cons}) = \mathsf{int}, \mathsf{list} \to \mathsf{list}$ $\quad a(\mathsf{car}) = \mathsf{list} \to \mathsf{int}$ $\quad a(\mathsf{cdr}) = \mathsf{list} \to \mathsf{list}$
$a(\mathsf{nil}) = \to \mathsf{list}$ (i.e. nil is a constant of sort list)
$a(b) = \to \mathsf{int}$ (i.e. b is a constant of sort int)
$a(p) = \mathsf{int}, \mathsf{list}$.

Let X_{int} be the set of variables of sort int containing $\{i, j, k\}$, and let X_{list} be the set of variables of sort list containing $\{x, y, z\}$. Let $X = \{X_{\mathsf{int}}, X_{\mathsf{list}}\}$. Which of the following expressions are terms over Σ and X, which are atoms/literals/clauses/formulae[1], which are neither?

(a) $\mathsf{cons}(\mathsf{cons}(b, \mathsf{nil}), \mathsf{nil})$
(b) $\mathsf{cons}(b, \mathsf{cons}(b, \mathsf{nil}))$
(c) $\neg p(b, \mathsf{cons}(b, \mathsf{cons}(b, \mathsf{nil})))$
(d) $\neg p(\mathsf{cons}(b, \mathsf{nil}), \mathsf{cons}(b, \mathsf{cons}(b, \mathsf{nil})))$
(e) $\mathsf{cons}(b, \mathsf{cons}(b, \mathsf{nil})) \approx_l \mathsf{cons}(\mathsf{cons}(x, b), \mathsf{nil})$
(f) $\mathsf{cons}(i, \mathsf{cons}(b, \mathsf{nil})) \approx j$
(g) $p(\neg \mathsf{car}(x), x)$
(h) $\neg p(\mathsf{car}(x), x) \vee p(j, \mathsf{cons}(j, x))$
(i) $\neg p(b, x) \vee p(b, \mathsf{cons}(b, x)) \vee b$
(j) $\forall i : \mathsf{int}, \forall x : \mathsf{list}\ (\mathsf{cons}(\mathsf{car}(x), \mathsf{cdr}(x)) \approx_l x)$
(k) $\exists i : \mathsf{int}, \forall y : \mathsf{list}\ (\mathsf{cons}(b, p(x, y)) \approx_l \mathsf{cdr}(y))$

You will be able to solve this exercise only after the terminology is introduced in the lecture on Monday, 21.11.2016.

[1] In first-order logic with equality, where equality between terms of sort int is $\approx_i$ and equality between terms of sort list is $\approx_l$.

Supplementary exercise (will be discussed in the exercise session)

Exercise 4.6: *(5 P)*
Let N be a set of clauses in propositional logic with the property that each clause consists of two literals. Prove that the satisfiability of N can be checked in polynomial time in the size of N.

Hint (way to a possible solution):

- How many clauses consisting of two literals (over a *finite* set of propositional variables $\Pi = \{P_1, \ldots, P_n\}$) exist?
- Analyze the form of possible resolution inferences from N.
- Let N be a set of clauses in propositional logic with the property that each clause consists of two literals. Show that
  - If N is satisfiable then we cannot generate from N, using the resolution calculus, both $P \vee P$ and $\neg P \vee \neg P$ for some propositional variable P.
  - If we cannot generate from N, using the resolution calculus, both $P \vee P$ and $\neg P \vee \neg P$ for some propositional variable P then N is satisfiable.
- Show that the number of inferences by resolution from N which yield different clauses is polynomial in the size of N and in the size of Π. Infer that the satisfiability of N can be checked in polynomial time in the size of N.

Please submit your solution until Wednesday, November 23, 2014 at 12:00. Joint solutions prepared by up to three persons are allowed. Please do not forget to write your name on your solution.

Submission possibilities:

- By e-mail to `sofronie@uni-koblenz.de` with the keyword “Homework DP” in the subject.
- Put it in the box in front of Room B 222.