

Exercise 5.4 Let $\Sigma = (\Sigma, \Pi)$ be a signature and X a set of variables. Let A be a Σ -structure and $\beta: X \rightarrow U_A$ a variable assignment.

(1)

- (1) Prove that for every formula $F \in F_\Sigma(x)$ and every $x \in X$ the truth value $A(\beta)(\forall x F)$ and $A(\beta)(\exists x F)$ do not depend on $\beta(x)$.

Proof: $A(\beta)(\forall x F) = \min_{a \in U_A} A(\beta[x \mapsto a])(F)$

$$A(\beta)(\exists x F) = \max_{a \in U_A} A(\beta[x \mapsto a])(F)$$

The values do not depend on $\beta(x)$.

- (2) Use (1) to show that if G is a closed formula in $F_\Sigma(x)$ then the truth value of G in A wrt β , $A(\beta)(G)$ does not depend on β .

Proof: We show that the following, stronger, statement holds, and that (2) follows from that statement.

Statement: Let G be a quantified formula (no variable occurs both free and bound; no variable is quantified more than once in G), let A be a structure and $\beta: X \rightarrow U_A$ a var. assignment. Assume that G has free variables x_1, \dots, x_n and bounded variables y_1, \dots, y_m .

Then the value $A(\beta)(G)$ does not depend on $\beta(y_1), \dots, \beta(y_m)$.

Proof of the statement: by structural induction.

Induction basis: $G = \top, G = \perp$ or G is an atom.

In all these cases G does not have bounded variables, so the statement holds.

Let G be a formula which is not \top, \perp or atomic.

Induction hypothesis: We assume that the statement holds for all strict subformulae of G .

Induction step: We prove that the statement holds then also for G .

We make a case distinction:

Case 1: $G = \forall G_1$. Then G and G_1 have the same bounded variables y_1, \dots, y_m .

By the induction hypothesis, $A(\beta)(G_1)$ does not depend on $\beta(y_1), \dots, \beta(y_m)$. Hence $A(\beta)(G)$ does not depend on $\beta(y_1), \dots, \beta(y_m)$.

Case 2: $G = G_1 \circ G_2$ $\circ \in \{ \wedge, \vee, \rightarrow, \leftrightarrow \}$. (2)

if y_1, \dots, y_n are bounded in G then some of them are bounded in G_1 and some are bounded in G_2

and as G is purified no free variable of G_1 occurs bounded in G_2 and no free variable of G_2 occurs bounded in G_1 .

let y_1, \dots, y_n be the bounded variables in G_1 ,
 y_{n+1}, \dots, y_n be the bounded variables in G_2 .

- By the induction hypothesis, $A(\beta)(G_1)$ does not depend on $\beta(y_1) \dots \beta(y_n)$.
- The variables y_1, \dots, y_n do not occur at all in G_2 , so $A(\beta)(G_2)$ does not depend on $\beta(y_1) \dots \beta(y_n)$.
- By the induction hypothesis, $A(\beta)(G_2)$ does not depend on $\beta(y_{n+1}) \dots \beta(y_n)$.
- The variables $y_{n+1} \dots y_n$ do not occur at all in G_1 , so $A(\beta)(G_1)$ does not depend on $\beta(y_{n+1}) \dots \beta(y_n)$.

In conclusion: $A(\beta)(G_1)$ does not depend on $\beta(y_1) \dots \beta(y_n)$
 $A(\beta)(G_2)$ does not depend on $\beta(y_1) \dots \beta(y_n)$

so $A(\beta)(G) = A(\beta)(G_1) \circ A(\beta)(G_2)$ does not depend on $\beta(y_1) \dots \beta(y_n)$.

Case 3: $G = \forall y, G_1$.

The bounded variables in G are $y_1, y_2 \dots y_n$, so

the bounded variables in G_1 are $y_2 \dots y_n$.

By the induction hypothesis, in every $\beta': X \rightarrow \cup A$,
 $A(\beta')(G_1)$ does not depend on $\beta'(y_2) \dots \beta'(y_n)$.

Then $A(\beta)(G) = A(\beta)(\forall y, G_1) = \min_{a \in \cup A} A(\beta[y \mapsto a])(G_1)$

$\underbrace{\qquad\qquad\qquad}_{\text{value does not depend}} \underbrace{\qquad\qquad\qquad}_{\text{on } \beta(y_2) \dots \beta(y_n)} \underbrace{\qquad\qquad\qquad}_{\text{value does not depend on } \beta(y_1)}$

→ the value

$A(\beta)(G)$ does not depend on $\beta(y_1) \dots \beta(y_n)$

Case 4: $G = \exists y, G_1$: analogous to Case 3.

(3)

Remark (2) follows from the statement.

Proof: Assume that for every A and β and every formula G , if G is purified then $A(\beta)(G)$ does not depend on value in β of the bounded variables in G .

Let G be a closed formula.

- We can w.l.o.g assume that G is purified (otherwise we can purify G and obtain an equivalent formula).
- All the variables in G (say $y_1 \dots y_n$) are bounded.
From the statement on p. 1 we know that $A(\beta)(G)$ does not depend on $\beta(y_1) \dots \beta(y_n)$.
- The other variables in x do not occur in G , so.
 $A(\beta)(G)$ does not depend on $\beta(x)$ for $x \notin \{y_1 \dots y_n\}$.

Thus $A(\beta)(G)$ does not depend on $\beta(x)$, for $x \in X$.

(3) Use (2) to prove that $\text{Th}(\text{Mod}(F)) = \{G \in F_\Sigma(x) \text{ closed} \mid FFG\}$

if F is a set of closed formulae

Proof: $\text{Th}(\text{Mod}(F)) = \{G \in F_\Sigma(x) \text{ closed} \mid \forall A \models G \text{ for all } A \in \text{Mod}(F)\}$

$$= \{G \in F_\Sigma(x) \text{ closed} \mid \text{if } \forall F \in F \text{ for all } F \in F \text{ then } A \models G\}.$$

We show that for every $G \in F_\Sigma(x)$ closed,

$G \in \text{Th}(\text{Mod}(F)) \iff FFG$.

Proof: \Rightarrow Assume $G \in \text{Th}(\text{Mod}(F))$.

Then for every Σ -structure A :

$\text{if } A \models F \text{ for all } F \in F \text{ then } A \models G$. (*)

We show FFG . Let A, β be such that $A(\beta)(F) = 1$ for all $F \in F$
we want to show that $A(\beta)(G) = 1$:

Since all formulae in F are closed, if $A(\beta)(F) = 1$ for some β
then $A(\beta)(F) = 1$ for all β . So if $A(\beta)(F) = 1$ for all $F \in F$
then $A \models F$ for all $F \in F$
By (*) it follows that $A \models G$,
so $A(\beta)(G) = 1$.

\Leftarrow " Assume $\mathcal{F} \models G$.

" Then for every Σ -structure A and every valuation $\beta: X \rightarrow \mathcal{U}_A$
if $A(\beta)(F) = 1$ for all $F \in \mathcal{F}$ **
then $A(\beta)(G) = 1$.

We show that $G \in \text{Th}(\text{Mod}(\mathcal{F}))$ i.e. \oplus .

| Let A be a Σ -structure such that $A \models F$ for all $F \in \mathcal{F}$.
| We show that $A \models G$, i.e. $A(\beta)(G) = 1$ for all $\beta: X \rightarrow \mathcal{U}_A$.

... let β be a valuation $\beta: X \rightarrow \mathcal{U}_A$
As $A \models F$ for all $F \in \mathcal{F}$ we have $A(\beta)(F) = 1$ for all $F \in \mathcal{F}$
From \oplus it then follows $A(\beta)(G) = 1$ qed.