

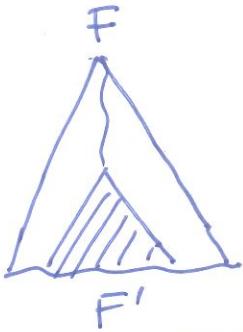
F formula , P new propositional variable
 F' subformula of F

(1)

$F[F']$: F with its subformula F' not replaced

$F[P]$: F with its subformula F' replaced by P.

TERMINOLOGY AND EXPLANATIONS

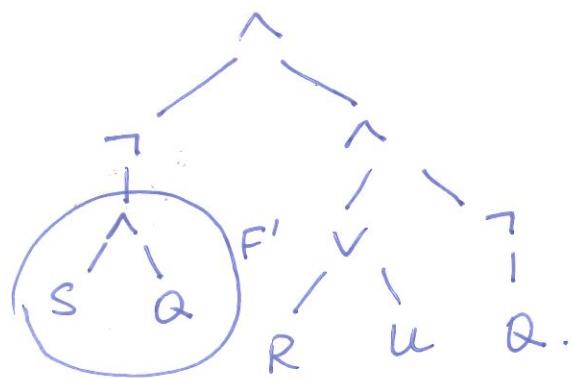


$F[F']$

The formula F
 (subformula F' not replaced).

Example : $F : \neg(S \wedge Q) \wedge ((R \vee U) \wedge \neg Q)$

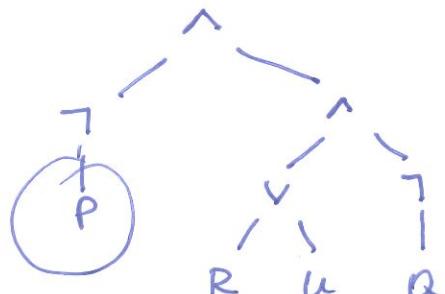
$F' : S \wedge Q$.



$F[F']$

P new propositional variable

$F[P]$ is, in this context,
 the formula obtained
 by replacing F' with P

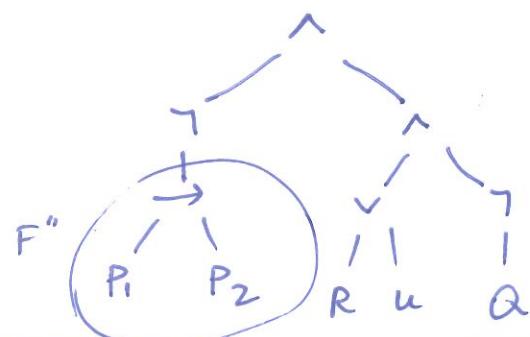


$F[P]$

F'' new formula.

$F[F'']$ is, in this context,
 the formula obtained
 by replacing F' with F'' .

$F'' = P_1 \rightarrow P_2$



$F[F'']$

Task: Prove: $F[F']$ satisfiable iff $F[P] \wedge (P \leftrightarrow F')$ satisfiable (2)

We prove first the following lemma:

Lemma 1: Let $\mathcal{A}: \Pi \rightarrow \{\top, \perp\}$ s.t. $\mathcal{A}(P) = \mathcal{A}(F')$. Then $\mathcal{A}(F[P]) = \mathcal{A}(F[F'])$.

$\mathcal{A}(F)$

Proof: Structural induction.

Induction basis: Let $F \in \{\top, \perp\} \cup \Pi$

Then F has only one subformula (itself)

Thus, $F = F'$, hence $F[F'] = F'$
 $F[P] = P$.

Then: $\mathcal{A}(F[P]) = \mathcal{A}(P) = \mathcal{A}(F') = \mathcal{A}(F[F'])$. qed.

Let F be a formula s.t. $F \notin \{\top, \perp\} \cup \Pi$.

Induction hypothesis: $\mathcal{P}(G)$ holds for all strict subformulae G of F

Induction step: We prove that $\mathcal{P}(F)$ holds.

We know that F' is a subformula of F .

Case 1: $F' = F$. Then $\begin{cases} F[F'] = F' \\ F[P] = P \end{cases}$

Hence, $\mathcal{A}(F[P]) = \mathcal{A}(P) = \mathcal{A}(F') = \mathcal{A}(F[F'])$.

Case 2: $F' \neq F$. We distinguish the following cases:

case 2.1: $F = \neg G$. Then F' is a subformula of G .

$$F[F'] = \neg G[F']$$

$$F[P] = \neg G[P].$$

ind hyp.

Then: $\mathcal{A}(F[P]) = \mathcal{A}(\neg G[P]) = \neg \mathcal{A}(G[P]) \stackrel{\downarrow}{=} \neg \mathcal{A}(G[F'])$
 $= \mathcal{A}(\neg G[F']) = \mathcal{A}(F[F'])$. qed.

Case 2.2: $F = G_1 \circ G_2$ $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

Then F' is a subformula of G_1 or of G_2 .

Assume F' is a subformula of G_1 (the other case is analogous)

$$F[F'] = G_1[F'] \circ G_2 \quad \left. \begin{array}{l} \dots \\ F[P] = G_1[P] \circ G_2 \end{array} \right\} \Rightarrow \mathcal{A}(F[P]) = \mathcal{A}(G_1[P] \circ G_2) = \mathcal{A}(G_1[P]) \circ \mathcal{A}(G_2) =$$

$$\mathcal{A}(G_1[F']) \circ \mathcal{A}(G_2) = \mathcal{A}(G_1[F'] \circ G_2) = \mathcal{A}(F[F'])$$

qed

Theorem 1: $F[P] \wedge (P \leftrightarrow F')$ satisfiable iff $F[F']$ satisfiable

Proof: " \Rightarrow " Assume $F[P] \wedge (P \leftrightarrow F')$ is satisfiable, i.e. there exists $A: \Pi \cup \{P\} \rightarrow \{0, 1\}$ with $\begin{cases} A(F[P]) = 1 \\ A(P \leftrightarrow F') = 1. \end{cases}$

Then $A(P) = A(F')$, hence by the previous lemma,

$$A(F[P]) = A(F[F']) = 1.$$

Thus, $F[F']$ is satisfiable.

" \Leftarrow " Assume $F[F']$ is satisfiable, i.e. there exists $A: \Pi \rightarrow \{0, 1\}$ with $A(F[F']) = 1$.

let $\bar{A}: \Pi \cup \{P\} \rightarrow \{0, 1\}$ be defined by:

$$\bar{A}(Q) = \begin{cases} A(Q) & \text{if } Q \in \Pi, Q \neq P \\ A(F') & \text{if } Q = P. \end{cases}$$

Since P does not occur in $F[F']$,

$$\bar{A}(F[F']) = A(F[F']) = 1 \quad (\text{easy proof by structural ind.})$$

Since $\bar{A}(P) = \bar{A}(F')$, we know that

$$1 = \bar{A}(F[F']) = \bar{A}(F[P]).$$

$$\begin{aligned} \text{Thus, } \bar{A}(F[P] \wedge (P \leftrightarrow F')) &= \bar{A}(F[P]) \wedge_b (\bar{A}(P) \leftrightarrow_b \bar{A}(F')) \\ &= 1. \end{aligned}$$

Thus $F[P] \wedge (P \leftrightarrow F')$ is satisfiable.

Lemma 2: Let F be a formula, F' a subformula of F and F'' be another formula.

(1) If F' has positive polarity in F :

if $(A \models F[F']) \text{ and } A(F') \leq A(F''))$ then $A \models F[F'']$.

(2) If F' has negative polarity in F :

if $(A \models F[F']) \text{ and } A(F'') \leq A(F'))$ then $A \models F[F'']$.

Proof: by structural induction.

Induction basis: Let $F \in \{\top, \perp\} \cup \Pi$. Then F has only itself as subformula.

$\Rightarrow \begin{cases} F' = F \text{ and } F' \text{ has positive polarity in } F \\ F[F'] = F', \quad F[F''] = F'' \text{ for every formula } F' \end{cases}$

Assume $A(F[F']) = 1$ and $A(F') \leq A(F'')$.

Then $A(F[F'']) = A(F'') \geq A(F') = A(F[F']) = 1$.

Hence $A \models F[F'']$.

Let F be a formula with $F \notin \{\top, \perp\} \cup \Pi$.

Induction hypothesis: (1) and (2) hold for all strict subformulas of F .

Induction step: We prove that (1) and (2) hold for F .

Let F' be a subformula of F .

Case 1: $F' = F$. Then (1) can be proved as in the induction basis.

Case 2: $F' \neq F$. We distinguish the following cases:

case 2.1: $F = \neg G$. Then F' is a subformula of G and $\begin{cases} F[F'] = \neg G[F'] \\ F[F''] = \neg G[F''] \end{cases}$

Subcase 2.1.a: F' has positive polarity in F .

Then F' has negative polarity in G , hence by the ind-hyp. (2) holds for G . We prove that (1) holds for F .

Assume $A(F[F']) = 1$ and $A(F') \leq A(F'')$.

Assume $A(F[F'']) = 0$. Then $A(\neg G[F'']) = 0$, so $A(G[F'']) = 1$.

As $A(F') \leq A(F'')$, by (2) it follows that $A(G[F']) = 1$ & Ind-hyp. (hence $A(\neg G[F']) = 0$).

This is a contradiction. Thus $A(F[F'']) = 1$. $A(F[F'])$

Subcase b): F' has negative Volatility in F $\Rightarrow F'$ has positive Volatility in G .

Induction hypothesis: (1) holds for G .

Prove (2) for F .

Let A be st. $A \in F[F']$ and $A(F'') \leq A(F')$

Assume $A(F[F'']) = 0$.

Then $A(7G[F'']) = 0$, so $A(0(F'')) = 1$.

$$A(7G[F']) = 0$$

$$\Rightarrow A(F[F']) = 0 \text{ almost.}$$

Case 2.2: $F = G_1 \wedge G_2$

Subcase a): F' vs. volatility of F

$\Rightarrow F'$ subfinsle with vs. volatility of G_i

$$i \in \{1, 2\}$$

ind. hypothesis: (1) holds for G_i

we prove (1) for F .

Let A be st. $A \in F[F']$ and $A(F') \leq A(F'')$

Then $A \in G_i$ for $i=1, 2$

$\Rightarrow A \in G_i[F'] \Rightarrow A \in G_i[F'']$ $\begin{cases} \Rightarrow A \in G_i[F''] \wedge G_j \\ j \neq i \end{cases}$

$$A(F') \leq A(F'')$$

$$\Rightarrow A \in F[F'']$$

Subcase b)

case when F' 's q. vol. in F similar.

Case 2.3: $F = G_1 \vee G_2$

analogous.

Theorem 2

F formula containing neither \rightarrow nor \leftrightarrow .

P new propositional variable, F' subformula of F .

(1) if F' has positive polarity in F then

$$F[F'] \text{ satisfiable} \iff F[P] \wedge (P \rightarrow F') \text{ satisfiable}$$

(2) if F' has negative polarity in F then

$$F[F'] \text{ satisfiable} \iff F[P] \wedge (F' \rightarrow P) \text{ satisfiable}$$

Proof

(1) Assume F' has positive polarity in F .

" \Rightarrow " Assume $F[F']$ satisfiable, i.e. there exists $A: \Pi \rightarrow \{0,1\}$ with $A(F[F']) = 1$.

Let $\bar{A}: \Pi \cup \{P\} \rightarrow \{0,1\}$ be defined by $\bar{A}(Q) = \begin{cases} A(Q) & Q \neq P \\ A(F') & Q = P \end{cases}$.

As in the proof of Theorem 1 we can show that $\bar{A}(F[P]) = A(F[F']) = 1$ and $\bar{A}(P \rightarrow F') = 1$, hence $\bar{A}(F[P] \wedge (P \rightarrow F')) = 1$, so $F[P] \wedge (P \rightarrow F')$ is satisfiable.

" \Leftarrow " Assume $F[P] \wedge (P \rightarrow F')$ is satisfiable, i.e. there exists $A: \Pi \cup \{P\} \rightarrow \{0,1\}$ with $\begin{cases} A(F[P]) = 1 \\ A(P) \leq A(F') \end{cases}$.

By Lemma 2(1), it follows that $A(F[F']) = 1$ qed.

(2) Assume F' has negative polarity in F .

" \Rightarrow " The proof is analogous to the proof in (1, \Rightarrow).

" \Leftarrow " Assume $F[P] \wedge (F' \rightarrow P)$ is satisfiable, i.e. there exists $A: \Pi \cup \{P\} \rightarrow \{0,1\}$ with $\begin{cases} A(F[P]) = 1 \\ A(F') \leq A(P) \end{cases}$.

By Lemma 2(2) it follows that $A(F[F'']) = 1$. qed