Decision Procedures for Verification

Part 1. Propositional Logic (2)

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Last time

Propositional Logic

1.1 Syntax

- Language
  - propositional variables
  - logical symbols
    $\Rightarrow$ Boolean combinations
- Propositional Formulae

1.2 Semantics

- Valuations
- Truth value of a formula in a valuation
- Models, Validity, and Satisfiability

Canonical forms

- CNF and DNF
- Computing CNF/DNF by rewriting the formulae
- Structure-Preserving Translation for CNF
- Optimized translation using polarity

Decision Procedures for Satisfiability

- Simple Decision Procedures

  truth table method

- The Resolution Procedure

- The Davis-Putnam-Logemann-Loveland Algorithm

1.5 Inference Systems and Proofs

Inference systems Γ (proof calculi) are sets of tuples

$$(F_1, \ldots, F_n, F_{n+1}), \;\; n \geq 0,$$

called inferences or inference rules, and written

$$\frac{\overbrace{F_1 \;\; \ldots \;\; F_n}^{\text{premises}}}{\underbrace{F_{n+1}}_{\text{conclusion}}} \;.$$

Clausal inference system: premises and conclusions are clauses. One also considers inference systems over other data structures.

Proofs

A proof in Γ of a formula F from a a set of formulas N (called assumptions) is a sequence $F_1, \ldots, F_k$ of formulas where

(i) $F_k = F$,

(ii) for all $1 \leq i \leq k$: $F_i \in N$, or else there exists an inference $(F_{i_1}, \ \ldots, \ F_{i_{n_i}}, \ F_i)$ in Γ, such that $0 \leq i_j < i$, for $1 \leq j \leq n_i$.

Soundness and Completeness

Provability $\vdash_\Gamma$ of F from N in Γ:

$N \vdash_\Gamma F \;\; :\Leftrightarrow$ there exists a proof Γ of F from N.

Γ is called sound $:\Leftrightarrow$

$$\frac{F_1 \;\; \ldots \;\; F_n}{F} \in \Gamma \quad \Rightarrow \quad F_1, \ldots, F_n \models F$$

Γ is called complete $:\Leftrightarrow$

$$N \models F \quad \Rightarrow \quad N \vdash_\Gamma F$$

Γ is called refutationally complete $:\Leftrightarrow$

$$N \models \bot \quad \Rightarrow \quad N \vdash_\Gamma \bot$$

1.6 The Propositional Resolution Calculus

Resolution inference rule:

$$\frac{C \vee A \qquad \neg A \vee D}{C \vee D}$$

Terminology: $C \vee D$: resolvent; A: resolved atom

(Positive) factorisation inference rule:

$$\frac{C \vee A \vee A}{C \vee A}$$

C, D: clauses

A atom (propositional variable)

The Resolution Calculus *Res*

These are schematic inference rules; for each substitution of the schematic variables C, D, and A, respectively, by propositional clauses and atoms we obtain an inference rule.

As "$\vee$" is considered associative and commutative, we assume that A and $\neg A$ can occur anywhere in their respective clauses.

Sample Refutation

1.	$\neg P \vee \neg P \vee Q$	(given)
2.	$P \vee Q$	(given)
3.	$\neg R \vee \neg Q$	(given)
4.	R	(given)
5.	$\neg P \vee Q \vee Q$	(Res. 2. into 1.)
6.	$\neg P \vee Q$	(Fact. 5.)
7.	$Q \vee Q$	(Res. 2. into 6.)
8.	Q	(Fact. 7.)
9.	$\neg R$	(Res. 8. into 3.)
10.	$\perp$	(Res. 4. into 9.)

Resolution with Implicit Factorization *RIF*

$$\frac{C \vee A \vee \ldots \vee A \qquad \neg A \vee D}{C \vee D}$$

1.	$\neg P \vee \neg P \vee Q$	(given)
2.	$P \vee Q$	(given)
3.	$\neg R \vee \neg Q$	(given)
4.	R	(given)
5.	$\neg P \vee Q \vee Q$	(Res. 2. into 1.)
6.	$Q \vee Q \vee Q$	(Res. 2. into 5.)
7.	$\neg R$	(Res. 6. into 3.)
8.	$\perp$	(Res. 4. into 7.)

Soundness of Resolution

Theorem 1.10. Propositional resolution is sound.

Proof:

Let $\mathcal{A}$ valuation. To be shown:

(i) for resolution: $\mathcal{A} \models C \vee A,\ \mathcal{A} \models D \vee \neg A \ \Rightarrow\ \mathcal{A} \models C \vee D$

(ii) for factorization: $\mathcal{A} \models C \vee A \vee A \ \Rightarrow\ \mathcal{A} \models C \vee A$

(i): Assume $\mathcal{A}^*(C \vee A) = 1, \mathcal{A}^*(D \vee \neg A) = 1$.
Two cases need to be considered: (a) $\mathcal{A}^*(A) = 1$, or (b) $\mathcal{A}^*(\neg A) = 1$.
(a) $\mathcal{A} \models A \Rightarrow \mathcal{A} \models D \Rightarrow \mathcal{A} \models C \vee D$
(b) $\mathcal{A} \models \neg A \Rightarrow \mathcal{A} \models C \Rightarrow \mathcal{A} \models C \vee D$

(ii): Assume $\mathcal{A} \models C \vee A \vee A$. Note that $\mathcal{A}^*(C \vee A \vee A) = \mathcal{A}^*(C \vee A)$, i.e. the conclusion is also true in $\mathcal{A}$.

Soundness of Resolution

Note: In propositional logic we have:

1. $\mathcal{A} \models L_1 \vee \ldots \vee L_n \;\Leftrightarrow\;$ there exists i: $\mathcal{A} \models L_i$.
2. $\mathcal{A} \models A$ or $\mathcal{A} \models \neg A$.

Completeness of Resolution

How to show refutational completeness of propositional resolution:

- We have to show: $N \models \bot \;\Rightarrow\; N \vdash_{Res} \bot$,

  or equivalently: If $N \not\vdash_{Res} \bot$, then N has a model.

- Idea: Suppose that we have computed sufficiently many inferences (and not derived $\bot$).

  Now order the clauses in N according to some appropriate ordering, inspect the clauses in ascending order, and construct a series of valuations.

- The limit valuation can be shown to be a model of N.

Clause Orderings

1. We assume that $\succ$ is any fixed ordering on propositional variables that is *total* and well-founded.

2. Extend $\succ$ to an ordering $\succ_L$ on literals:

$$
\begin{array}{lcll}
[\neg]P & \succ_L & [\neg]Q & \text{, if } P \succ Q \\
\neg P & \succ_L & P &
\end{array}
$$

3. Extend $\succ_L$ to an ordering $\succ_C$ on clauses:
   $\succ_C = (\succ_L)_{\mathrm{mul}}$, the multi-set extension of $\succ_L$.

   Notation: $\succ$ also for $\succ_L$ and $\succ_C$.

Multi-Set Orderings

Let $(M, \succ)$ be a partial ordering. The multi-set extension of $\succ$ to multi-sets over M is defined by

$$\begin{aligned} S_1 \succ_{\mathrm{mul}} S_2 :\Leftrightarrow\ & S_1 \neq S_2 \\ & \text{and } \forall m \in M : [S_2(m) > S_1(m) \\ & \quad \Rightarrow \ \exists m' \in M : (m' \succ m \text{ and } S_1(m') > S_2(m'))] \end{aligned}$$

Theorem 1.11:

a) $\succ_{\mathrm{mul}}$ is a partial ordering.

b) $\succ$ well-founded $\Rightarrow$ $\succ_{\mathrm{mul}}$ well-founded

c) $\succ$ total $\Rightarrow$ $\succ_{\mathrm{mul}}$ total

Proof:

see Baader and Nipkow, page 22–24.

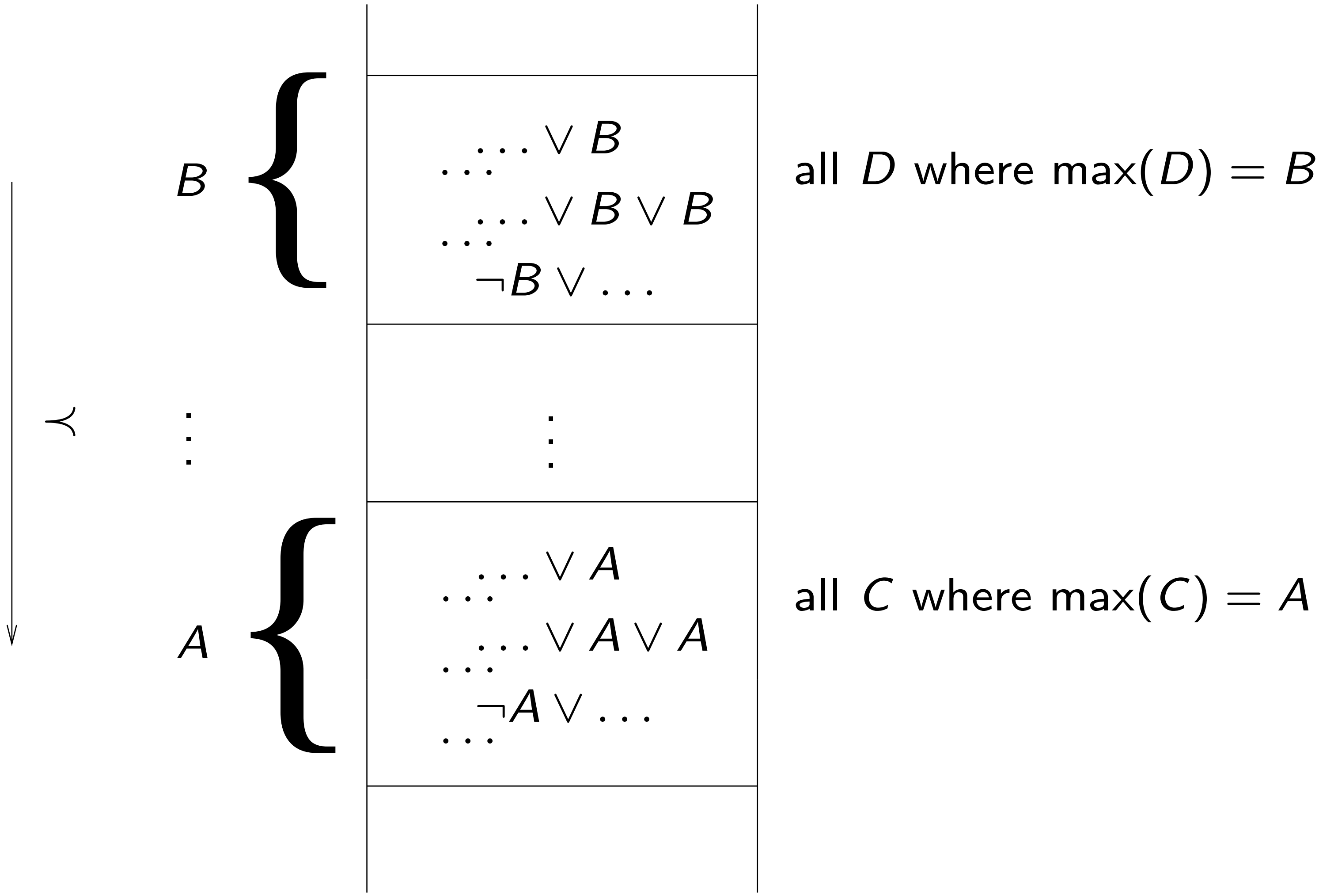
Example

Suppose $P_5 \succ P_4 \succ P_3 \succ P_2 \succ P_1 \succ P_0$. Then:

$$
\begin{array}{rc}
 & P_0 \vee P_1 \\
\prec & P_1 \vee P_2 \\
\prec & \neg P_1 \vee P_2 \\
\prec & \neg P_1 \vee P_4 \vee P_3 \\
\prec & \neg P_1 \vee \neg P_4 \vee P_3 \\
\prec & \neg P_5 \vee P_5
\end{array}
$$

Stratified Structure of Clause Sets

Let $A \succ B$. Clause sets are then stratified in this form:



Closure of Clause Sets under *Res*

$$Res(N) = \{C \mid C \text{ is concl. of a rule in } Res \text{ w/ premises in } N\}$$
$$Res^0(N) = N$$
$$Res^{n+1}(N) = Res(Res^n(N)) \cup Res^n(N), \text{ for } n \geq 0$$
$$Res^*(N) = \textstyle\bigcup_{n\geq 0} Res^n(N)$$

N is called saturated (wrt. resolution), if $Res(N) \subseteq N$.

Proposition 1.12

(i) $Res^*(N)$ is saturated.

(ii) *Res* is refutationally complete, iff for each set N of ground clauses:

$$N \models \bot \;\Leftrightarrow\; \bot \in Res^*(N)$$

Construction of Interpretations

Given: set N of clauses, atom ordering $\succ$.

Wanted: Valuation $\mathcal{A}$ such that

- "many" clauses from N are valid in $\mathcal{A}$;
- $\mathcal{A} \models N$, *if* N *is saturated and* $\bot \notin N$.

Construction according to $\succ$, starting with the minimal clause.

Main Ideas of the Construction

- Clauses are considered in the order given by $\prec$. We construct a model for N incrementally.
- When considering C, one already has a partial interpretation I_C (initially $I_C = \emptyset$) available.

  In what follows, instead of referring to partial valuations $\mathcal{A}_C$ we will refer to partial interpretations I_C (the set of atoms which are true in the valuation $\mathcal{A}_C$).

- If C is true in the partial interpretation I_C, nothing is done. ($\Delta_C = \emptyset$).
- If C is false, one would like to change I_C such that C becomes true.

Example

Let $P_5 \succ P_4 \succ P_3 \succ P_2 \succ P_1 \succ P_0$ (max. literals in red)

Construction of I:

	clauses C	I_C	Δ_C	Remarks
1	$\neg P_0$	$\emptyset$	$\emptyset$	true in $\mathcal{A}_C$
2	$P_0 \vee P_1$	$\emptyset$	$\{P_1\}$	
3	$P_1 \vee P_2$	$\{P_1\}$	$\emptyset$	true in $\mathcal{A}_C$
4	$\neg P_1 \vee P_2$	$\{P_1\}$	$\{P_2\}$	
5	$\neg P_1 \vee \neg P_1 \vee P_3 \vee P_0$	$\{P_1, P_2\}$	$\{P_3\}$	
6	$\neg P_1 \vee \neg P_1 \vee P_3 \vee P_3 \vee P_0$	$\{P_1, P_2, P_3\}$	$\emptyset$	true in $\mathcal{A}_C$
7	$\neg P_1 \vee P_4 \vee P_3 \vee P_0$	$\{P_1, P_2, P_3\}$	$\emptyset$	true in $\mathcal{A}_C$
8	$\neg P_1 \vee \neg P_4 \vee P_3$	$\{P_1, P_2, P_3\}$	$\emptyset$	true in $\mathcal{A}_C$
9	$\neg P_3 \vee P_5$	$\{P_1, P_2, P_3\}$	$\{P_5\}$	

The resulting $I = \{P_1, P_2, P_3, P_5\}$ is a model of the clause set.

Example

Let $P_5 \succ P_4 \succ P_3 \succ P_2 \succ P_1 \succ P_0$ (max. literals in red)

	clauses C	$I_C = \mathcal{A}_C^{-1}(1)$	Δ_C	Remarks
1	$\neg P_0$			
2	$P_0 \vee P_1$			
3	$P_1 \vee P_2$			
4	$\neg P_1 \vee P_2$			
5	$\neg P_1 \vee P_4 \vee P_3 \vee P_0$			
6	$\neg P_1 \vee \neg P_4 \vee P_3$			
7	$\neg P_1 \vee P_5$			

Example

Let $P_5 \succ P_4 \succ P_3 \succ P_2 \succ P_1 \succ P_0$ (max. literals in red)

	clauses C	$I_C = \mathcal{A}_C^{-1}(1)$	Δ_C	Remarks
1	$\neg P_0$	$\emptyset$	$\emptyset$	true in $\mathcal{A}_C$
2	$P_0 \vee P_1$			
3	$P_1 \vee P_2$			
4	$\neg P_1 \vee P_2$			
5	$\neg P_1 \vee P_4 \vee P_3 \vee P_0$			
6	$\neg P_1 \vee \neg P_4 \vee P_3$			
7	$\neg P_1 \vee P_5$			

Example

Let $P_5 \succ P_4 \succ P_3 \succ P_2 \succ P_1 \succ P_0$ (max. literals in red)

	clauses C	$I_C = \mathcal{A}_C^{-1}(1)$	Δ_C	Remarks
1	$\neg P_0$	$\emptyset$	$\emptyset$	true in $\mathcal{A}_C$
2	$P_0 \vee P_1$	$\emptyset$	$\{P_1\}$	P_1 maximal
3	$P_1 \vee P_2$			
4	$\neg P_1 \vee P_2$			
5	$\neg P_1 \vee P_4 \vee P_3 \vee P_0$			
6	$\neg P_1 \vee \neg P_4 \vee P_3$			
7	$\neg P_1 \vee P_5$			

Example

Let $P_5 \succ P_4 \succ P_3 \succ P_2 \succ P_1 \succ P_0$ (max. literals in red)

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2	$P_0 \vee P_1$	$\emptyset$	$\{P_1\}$	P_1 maximal
3	$P_1 \vee P_2$	$\{P_1\}$	$\emptyset$	true in $\mathcal{A}_C$
4	$\neg P_1 \vee P_2$			
5	$\neg P_1 \vee P_4 \vee P_3 \vee P_0$			
6	$\neg P_1 \vee \neg P_4 \vee P_3$			
7	$\neg P_1 \vee P_5$			

Example

Let $P_5 \succ P_4 \succ P_3 \succ P_2 \succ P_1 \succ P_0$ (max. literals in red)

	clauses C	$I_C = \mathcal{A}_C^{-1}(1)$	Δ_C	Remarks
1	$\neg P_0$	$\emptyset$	$\emptyset$	true in $\mathcal{A}_C$
2	$P_0 \vee P_1$	$\emptyset$	$\{P_1\}$	P_1 maximal
3	$P_1 \vee P_2$	$\{P_1\}$	$\emptyset$	true in $\mathcal{A}_C$
4	$\neg P_1 \vee P_2$	$\{P_1\}$	$\{P_2\}$	P_2 maximal
5	$\neg P_1 \vee P_4 \vee P_3 \vee P_0$			
6	$\neg P_1 \vee \neg P_4 \vee P_3$			
7	$\neg P_1 \vee P_5$			

Example

Let $P_5 \succ P_4 \succ P_3 \succ P_2 \succ P_1 \succ P_0$ (max. literals in red)

	clauses C	$I_C = \mathcal{A}_C^{-1}(1)$	Δ_C	Remarks
1	$\neg P_0$	$\emptyset$	$\emptyset$	true in $\mathcal{A}_C$
2	$P_0 \vee P_1$	$\emptyset$	$\{P_1\}$	P_1 maximal
3	$P_1 \vee P_2$	$\{P_1\}$	$\emptyset$	true in $\mathcal{A}_C$
4	$\neg P_1 \vee P_2$	$\{P_1\}$	$\{P_2\}$	P_2 maximal
5	$\neg P_1 \vee P_4 \vee P_3 \vee P_0$	$\{P_1, P_2\}$	$\{P_4\}$	P_4 maximal
6	$\neg P_1 \vee \neg P_4 \vee P_3$			
7	$\neg P_1 \vee P_5$			

Example

Let $P_5 \succ P_4 \succ P_3 \succ P_2 \succ P_1 \succ P_0$ (max. literals in red)

	clauses C	$I_C = \mathcal{A}_C^{-1}(1)$	Δ_C	Remarks
1	$\neg P_0$	$\emptyset$	$\emptyset$	true in $\mathcal{A}_C$
2	$P_0 \vee P_1$	$\emptyset$	$\{P_1\}$	P_1 maximal
3	$P_1 \vee P_2$	$\{P_1\}$	$\emptyset$	true in $\mathcal{A}_C$
4	$\neg P_1 \vee P_2$	$\{P_1\}$	$\{P_2\}$	P_2 maximal
5	$\neg P_1 \vee P_4 \vee P_3 \vee P_0$	$\{P_1, P_2\}$	$\{P_4\}$	P_4 maximal
6	$\neg P_1 \vee \neg P_4 \vee P_3$	$\{P_1, P_2, P_4\}$	$\emptyset$	P_3 not maximal; *min. counter-ex.*
7	$\neg P_1 \vee P_5$	$\{P_1, P_2, P_4\}$	$\{P_5\}$	

$I = \{P_1, P_2, P_4, P_5\} = \mathcal{A}^{-1}(1)$: $\mathcal{A}$ is not a model of the clause set

$\Rightarrow$ there exists a counterexample.

Main Ideas of the Construction

- Clauses are considered in the order given by $\prec$.
- When considering C, one already has a partial interpretation I_C (initially $I_C = \emptyset$) available.
- If C is true in the partial interpretation I_C, nothing is done. ($\Delta_C = \emptyset$).
- If C is false, one would like to change I_C such that C becomes true.

Main Ideas of the Construction

- Changes should, however, be *monotone*. One never deletes anything from I_C and the truth value of clauses smaller than C should be maintained the way it was in I_C.
- Hence, one chooses $\Delta_C = \{A\}$ if, and only if, C is false in I_C, if A occurs positively in C (*adding A will make C become true*) and if this occurrence in C is strictly maximal in the ordering on literals (*changing the truth value of A has no effect on smaller clauses*).

Resolution Reduces Counterexamples

$$\frac{\neg P_1 \vee P_4 \vee P_3 \vee P_0 \qquad \neg P_1 \vee \neg P_4 \vee P_3}{\neg P_1 \vee \neg P_1 \vee P_3 \vee P_3 \vee P_0}$$

Construction of I for the extended clause set:

	clauses C	I_C	Δ_C	Remarks
1	$\neg P_0$	$\emptyset$	$\emptyset$	
2	$P_0 \vee P_1$	$\emptyset$	$\{P_1\}$	
3	$P_1 \vee P_2$	$\{P_1\}$	$\emptyset$	
4	$\neg P_1 \vee P_2$	$\{P_1\}$	$\{P_2\}$	
8	$\neg P_1 \vee \neg P_1 \vee P_3 \vee P_3 \vee P_0$	$\{P_1, P_2\}$	$\emptyset$	P_3 occurs twice *minimal counter-ex.*
5	$\neg P_1 \vee P_4 \vee P_3 \vee P_0$	$\{P_1, P_2\}$	$\{P_4\}$	
6	$\neg P_1 \vee \neg P_4 \vee P_3$	$\{P_1, P_2, P_4\}$	$\emptyset$	old counterexample
7	$\neg P_1 \vee P_5$	$\{P_1, P_2, P_4\}$	$\{P_5\}$	

The same I, but smaller counterexample, hence some progress was made.

Factorization Reduces Counterexamples

$$\frac{\neg P_1 \vee \neg P_1 \vee P_3 \vee P_3 \vee P_0}{\neg P_1 \vee \neg P_1 \vee P_3 \vee P_0}$$

Construction of I for the extended clause set:

	clauses C	I_C	Δ_C	Remarks
1	$\neg P_0$	$\emptyset$	$\emptyset$	
2	$P_0 \vee P_1$	$\emptyset$	$\{P_1\}$	
3	$P_1 \vee P_2$	$\{P_1\}$	$\emptyset$	
4	$\neg P_1 \vee P_2$	$\{P_1\}$	$\{P_2\}$	
9	$\neg P_1 \vee \neg P_1 \vee P_3 \vee P_0$	$\{P_1, P_2\}$	$\{P_3\}$	
8	$\neg P_1 \vee \neg P_1 \vee P_3 \vee P_3 \vee P_0$	$\{P_1, P_2, P_3\}$	$\emptyset$	true in $\mathcal{A}_C$
5	$\neg P_1 \vee P_4 \vee P_3 \vee P_0$	$\{P_1, P_2, P_3\}$	$\emptyset$	
6	$\neg P_1 \vee \neg P_4 \vee P_3$	$\{P_1, P_2, P_3\}$	$\emptyset$	true in $\mathcal{A}_C$
7	$\neg P_3 \vee P_5$	$\{P_1, P_2, P_3\}$	$\{P_5\}$	

The resulting $I = \{P_1, P_2, P_3, P_5\}$ is a model of the clause set.

Construction of Candidate Models Formally

Let $N, \succ$ be given. We define sets I_C and Δ_C for all ground clauses C over the given signature inductively over $\succ$:

$$I_C := \bigcup_{C \succ D} \Delta_D$$

$$\Delta_C := \begin{cases} \{A\}, & \text{if } C \in N,\ C = C' \vee A,\ A \succ C',\ I_C \not\models C \\ \emptyset, & \text{otherwise} \end{cases}$$

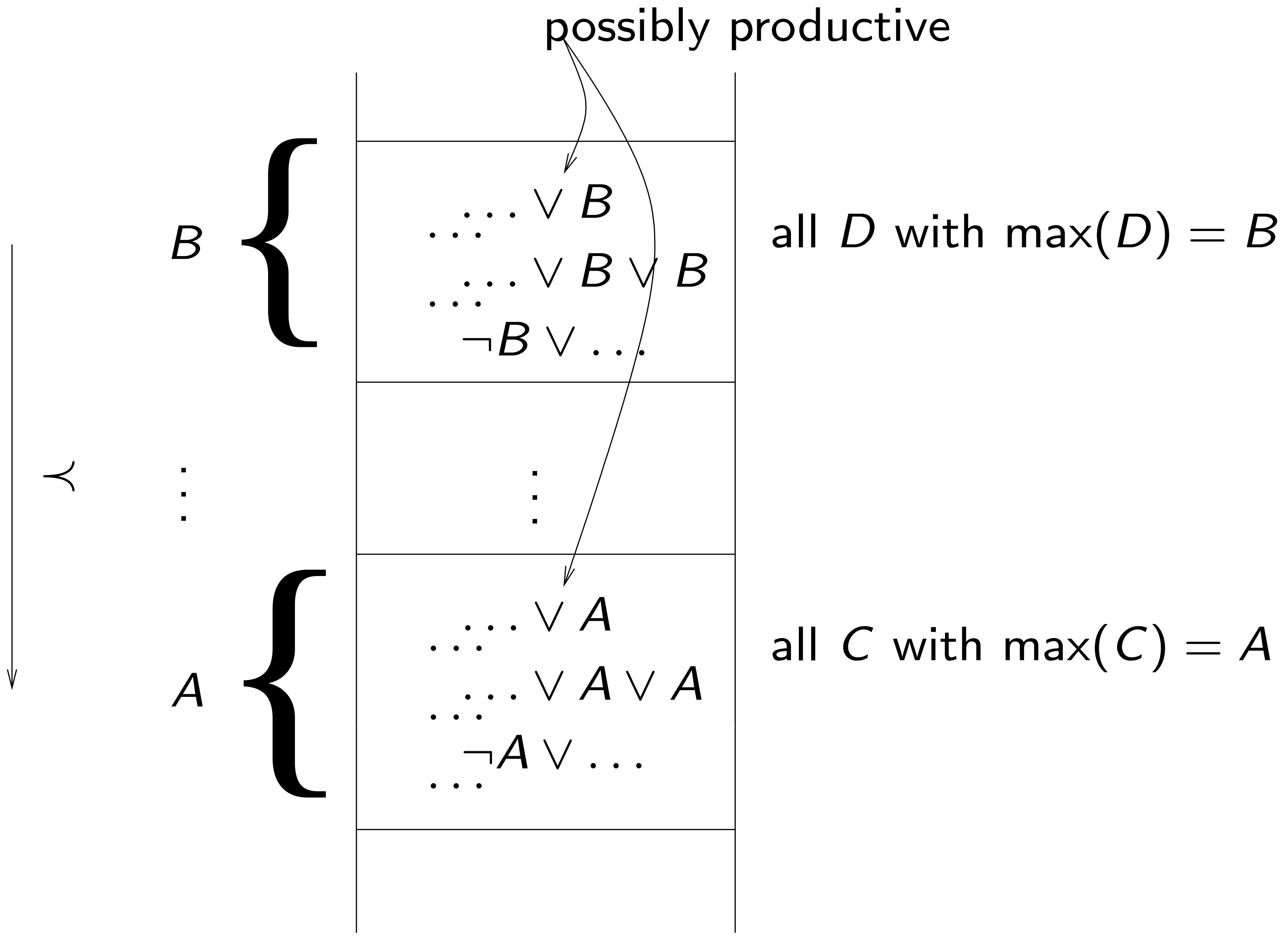
We say that C produces A, if $\Delta_C = \{A\}$.

The candidate model for N (wrt. $\succ$) is given as $I_N^{\succ} := \bigcup_C \Delta_C$.

We also simply write I_N, or I, for $I_N^{\succ}$ if $\succ$ is either irrelevant or known from the context.

Structure of $N, \succ$

Let $A \succ B$; producing a new atom does not affect smaller clauses.



Some Properties of the Construction

Proposition 1.13:

(i) $C = \neg A \vee C' \;\Rightarrow\;$ no $D \succeq C$ produces A.

(ii) C productive $\Rightarrow I_C \cup \Delta_C \models C$.

(iii) Let $D' \succ D \succeq C$. Then

$$I_D \cup \Delta_D \models C \Rightarrow I_{D'} \cup \Delta_{D'} \models C \text{ and } I_N \models C.$$

If, in addition, $C \in N$ or $\max(D) \succ \max(C)$:

$$I_D \cup \Delta_D \not\models C \Rightarrow I_{D'} \cup \Delta_{D'} \not\models C \text{ and } I_N \not\models C.$$

Some Properties of the Construction

(iv) Let $D' \succ D \succ C$. Then

$$I_D \models C \Rightarrow I_{D'} \models C \text{ and } I_N \models C.$$

If, in addition, $C \in N$ or $\max(D) \succ \max(C)$:

$$I_D \not\models C \Rightarrow I_{D'} \not\models C \text{ and } I_N \not\models C.$$

(v) $D = C \vee A$ produces $A \Rightarrow I_N \not\models C$.

Model Existence Theorem

Theorem 1.14 (Bachmair & Ganzinger):
Let $\succ$ be a clause ordering, let N be saturated wrt. Res, and suppose that $\bot \notin N$. Then $I_N^{\succ} \models N$.

Corollary 1.15:
Let N be saturated wrt. Res. Then $N \models \bot \Leftrightarrow \bot \in N$.

Model Existence Theorem

Proof:

Suppose $\bot \not\in N$, but $I_N^{\succ} \not\models N$. Let $C \in N$ minimal (in $\succ$) such that $I_N^{\succ} \not\models C$. Since C is false in I_N, C is not productive. As $C \neq \bot$ there exists a maximal atom A in C.

Case 1: $C = \neg A \vee C'$ (i.e., the maximal atom occurs negatively)
$\Rightarrow I_N \models A$ and $I_N \not\models C'$
$\Rightarrow$ some $D = D' \vee A \in N$ produces A. As $\frac{D' \vee A \qquad \neg A \vee C'}{D' \vee C'}$, we infer that $D' \vee C' \in N$, and $C \succ D' \vee C'$ and $I_N \not\models D' \vee C'$
$\Rightarrow$ contradicts minimality of C.

Case 2: $C = C' \vee A \vee A$. Then $\frac{C' \vee A \vee A}{C' \vee A}$ yields a smaller counterexample $C' \vee A \in N$. $\Rightarrow$ contradicts minimality of C.

Ordered Resolution with Selection

Ideas for improvement:

1. In the completeness proof (Model Existence Theorem) one only needs to resolve and factor maximal atoms
   $\Rightarrow$ if the calculus is restricted to inferences involving maximal atoms, the proof remains correct
   $\Rightarrow$ *order restrictions*

2. In the proof, it does not really matter with which negative literal an inference is performed
   $\Rightarrow$ choose a negative literal don't-care-nondeterministically
   $\Rightarrow$ *selection*

Selection Functions

A selection function is a mapping

$$S : C \quad \mapsto \quad \text{set of occurrences of } \textit{negative} \text{ literals in } C$$

Example of selection with selected literals indicated as $\boxed{X}$:

$$\boxed{\neg A} \vee \neg A \vee B$$

$$\boxed{\neg B_0} \vee \boxed{\neg B_1} \vee A$$

Ordered resolution

In the completeness proof, we talk about (strictly) maximal literals of clauses.

Resolution Calculus $Res_S^{\succ}$

Ordered Resolution with Selection:

$$\frac{C \vee A \qquad D \vee \neg A}{C \vee D}$$

if (i) $A \succ C$;

(ii) nothing is selected in C by S;

(iii) $\neg A$ is selected in $D \vee \neg A$,

or else nothing is selected in $D \vee \neg A$ and $\neg A \succeq \max(D)$.

Ordered Factoring:

$$\frac{C \vee A \vee A}{(C \vee A)}$$

if A is maximal in C and nothing is selected in C.

Note: For positive literals, $A \succ C$ is the same as $A \succ \max(C)$.

Search Spaces Become Smaller

1	$A \vee B$	
2	$A \vee \boxed{\neg B}$	
3	$\neg A \vee B$	
4	$\neg A \vee \boxed{\neg B}$	
5	$B \vee B$	Res 1, 3
6	B	Fact 5
7	$\neg A$	Res 6, 4
8	A	Res 6, 2
9	$\perp$	Res 8, 7

we assume $A \succ B$ and S as indicated by $\boxed{X}$. The maximal literal in a clause is depicted in red.

With this ordering and selection function the refutation proceeds strictly deterministically in this example. Generally, proof search will still be non-deterministic but the search space will be much smaller than with unrestricted resolution.