Decision Procedures for Verification

Part 1. Propositional Logic (4)

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Last time

Propositional Logic

Syntax

Semantics

Canonical forms

- Computing CNF/DNF by rewriting the formulae
- Structure-Preserving Translation for CNF
- Optimized translation using polarity

Decision Procedures for Satisfiability

- Simple Decision Procedures

  truth table method

- The Resolution Procedure
- The Davis-Putnam-Logemann-Loveland Algorithm

Today

- Applications of propositional logic

- First-order logic.

Applications of propositional logic

- A toy example (sudoku)
- Scheduling
- Verification

Sudoku

1								1	
2	4								
3		2							
4					5		4		7
5			8				3		
6			1		9				
7	3			4			2		
8		5		1					
9				8		6			

Idea: $p^d_{i,j} = \text{true}$ iff the value of square i, j is d

For example: $p^8_{3,5} = \text{true}$

Sudoku

Coding SUDOKU by propositional clauses:

1								1	
2	4								
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5			8				3		
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7	3			4			2		
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9				8		6			

- Concrete values result in units: $p^d_{i,j}$.
- For every value, column we generate: $\neg p^d_{i,j} \vee \neg p^d_{i,k}$ (if $j \neq k$).
  Accordingly for all rows and 3×3 boxes.
- For every square we generate: $p^1_{i,j} \vee \ldots p^9_{i,j}$.
  For every two different values d, d', and every square we generate: $\neg p^d_{i,j} \vee \neg p^{d'}_{i,j}$.
- For every value d and every column we generate:
  $p^d_{i,1} \vee \ldots p^d_{i,9}$.
  Accordingly for all rows and 3×3 boxes.

Sudoku

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3		2							
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9				8		6			

Set of clauses satisfiable $\Leftrightarrow$ Sudoku has a solution

Let $\mathcal{A}$ be a satisfying assignment

$\mathcal{A}(p_{i,j}^k) = 1$ iff a k appears in line i, column j.

Scheduling

Example: A simple scheduling problem

In a school there are three teachers with the following specialization combinations:

Müller	Mathematics
Schmidt	German
Körner	Mathematics, German

	Group a	Group b
8:00– 8:50	Mathematics	German
9:00– 9:50	German	German
10:00–10:50	Math	Mathematics

Each teacher must teach at least two classes.

Scheduling

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1) 8:00– 8:50	Mathematics	German
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Modeling:

Propositional variables: $P_{s,k,N,f}$ 'Teacher N teaches subject f in group k in time slot s'

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Modeling:

Propositional variables: $P_{s,k,N,f}$ 'Teacher N teaches subject f in group k in time slot s'

Rules: $(P_{1,a,M,m} \vee P_{1,a,K,m}) \wedge (P_{1,b,S,d} \vee P_{1,b,K,d})$

$(P_{2,a,S,d} \vee P_{2,a,K,d}) \wedge (P_{2,b,S,d} \vee P_{2,b,K,d})$

$(P_{3,a,M,m} \vee P_{3,a,K,m}) \wedge (P_{3,b,S,d} \vee P_{3,a,K,d})$

$\neg(P_{1,a,K,m} \wedge P_{1,b,K,d}) \wedge \neg(P_{2,a,K,d} \wedge P_{2,b,K,d}) \wedge \neg(P_{2,a,S,d} \wedge P_{2,b,S,d}) \wedge$

$\neg(P_{3,a,K,m} \wedge P_{3,b,K,m}) \wedge (P_{1,a,M,m} \wedge P_{1,b,M,m}) \ldots$

Program Verification

- Bounded model checking
- Model checking
- Invariant checking/generation
- Abstraction

Finite-state systems

- X finite set of variables, V finite set of possible values for the variables p^i_{xv} (in the i-th step x takes value v)
- Other propositional variables q_k, $k \in K$
- Transitions (variables change their value)

  $$Tr(i, i+1) := \bigvee \left(\text{Cond}(p^i_{x_1 v^i_1}, \dots, p^i_{x_n v^i_n}) \wedge \bigwedge_{j=1}^n p^{i+1}_{x_j v^{i+1}_j} \wedge \bigwedge_k q^{i+1}_k \right)$$

  (where v^{i+1}_j, q^{i+1}_k suitably computed)

$F(p^k_{x_1, v^k_1}, \dots, p^k_{x_n, v^k_n}, ...)$ property of assignments

Bounded model checking:

$$\bigwedge_{j=1}^{n} p^1_{x_j, v_j} \wedge \bigwedge q^1_k \wedge Tr(1, 2) \wedge \dots \wedge Tr(k-1, k) \wedge \neg F(p^k_{x_1, v^k_1}, \dots, p^k_{x_n, v^k_n}, ...)$$

Example

Question: Does BubbleSort return a sorted array?

int [] BubbleSort(int[] a) {
  int i, j, t;
  for ($i := |a| - 1; i > 0; i := i - 1$) {
    for ($j := 0; j < i; j := j + 1$) {
      if ($a[j] > a[j + 1]$){$t := a[j]$;
        $a[j] := a[j + 1]$;
        $a[j + 1] := t$};
}} return a}

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        for (j := 0; j < i; j := j + 1) {
            if (a[j] > a[j + 1]){t := a[j];
                                 a[j] := a[j + 1];
                                 a[j + 1] := t};
}} return a}
```

Simpler question:

$|a| = 3$; $a[0]=7$, $a[1]=9$, $a[2]=4$

does BubbleSort applied to this array return a sorted array?

Encoding in propositional logic:

- p_{ij}^{k} (at step k, $a[i] = k$)

  Examples: $p_{07}^{1}, p_{19}^{1}, p_{24}^{1}$

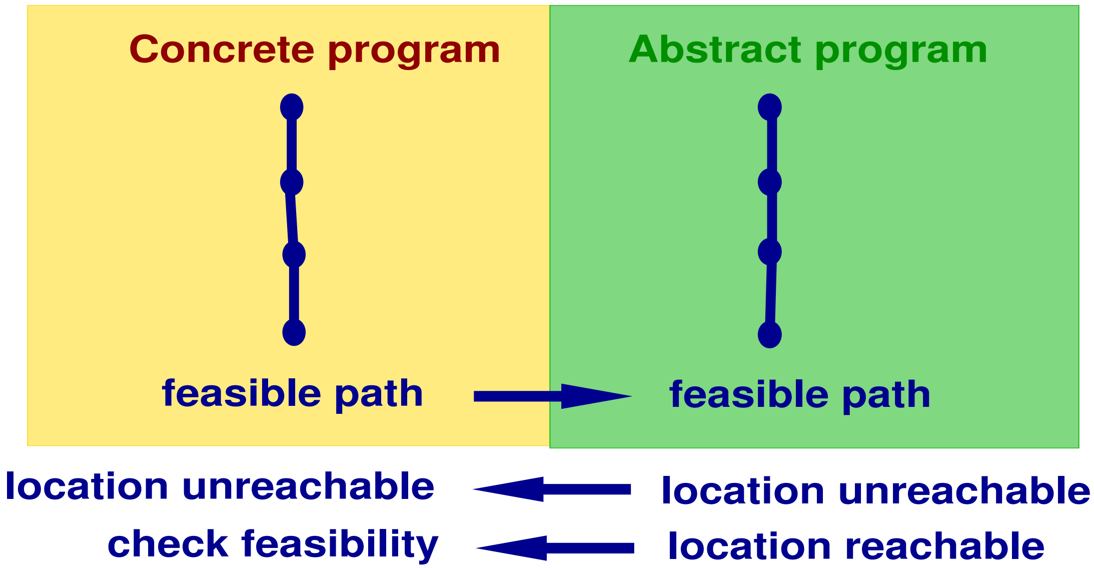
- gt_{ij}^{k} (at step k, $a[i] > a[j]$)

  Examples: $gt_{10}^{1}, \neg gt_{01}^{1}, gt_{02}^{1}, \neg gt_{20}^{1}, \dots$

Model updates with new propositional variables

(complicated; not very expressive)

Abstraction-Based Verification



$\Downarrow$

conjunction of constraints: $\phi(1) \wedge Tr(1,2) \wedge \cdots \wedge Tr(n-1,n) \wedge \neg \text{safe}(n)$

- satisfiable: feasible path
- unsatisfiable: refine abstract program s.t. the path is not feasible

Tools for SAT checking

http://www.satcompetition.org/

Examples of SAT solvers:

MiniSat: http://minisat.se/

MathSAT: http://mathsat.fbk.eu/publications.html (much more)

zChaff: http://www.princeton.edu/ chaff/zchaff.html

Example of use

Tools for SAT checking

Resolution-based theorem provers:

E: `http://www4.informatik.tu-muenchen.de/ schulz/E/E.html`

SPASS: `http://www.spass-prover.org/`

Vampire: `http://www.vprover.org/`

. . . full power for first-order logic (with equality)

Part 2: First-Order Logic

First-order logic

- formalizes fundamental mathematical concepts
- is expressive (Turing-complete)
- is not too expressive
  (e. g. not axiomatizable: natural numbers, uncountable sets)
- has a rich structure of decidable fragments
- has a rich model and proof theory

First-order logic is also called (first-order) predicate logic.

2.1 Syntax

Syntax:

- non-logical symbols (domain-specific)
  $\Rightarrow$ terms, atomic formulas
- logical symbols (domain-independent)
  $\Rightarrow$ Boolean combinations, quantifiers

Signature

A signature

$$\Sigma = (\Omega, \Pi),$$

fixes an alphabet of non-logical symbols, where

- Ω is a set of function symbols f with arity $n \geq 0$, written f/n,
- Π is a set of predicate symbols p with arity $m \geq 0$, written p/m.

If $n = 0$ then f is also called a constant (symbol).
If $m = 0$ then p is also called a propositional variable.
We use letters P, Q, R, S, to denote propositional variables.

Signature

Refined concept for practical applications:
many-sorted signatures (corresponds to simple type systems in programming languages).

Most results established for one-sorted signatures extend in a natural way to many-sorted signatures.

Many-sorted Signature

A many-sorted signature

$$\Sigma = (S, \Omega, \Pi),$$

fixes an alphabet of non-logical symbols, where

- S is a set of sorts,
- Ω is a set of function symbols f with arity $a(f) = s_1 \ldots s_n \to s$,
- Π is a set of predicate symbols p with arity $a(p) = s_1 \ldots s_m$

where $s_1, \ldots, s_n, s_m, s$ are sorts.

Variables

Predicate logic admits the formulation of abstract, schematic assertions. (Object) variables are the technical tool for schematization.

We assume that

$$X$$

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We assume that

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Many-sorted case:

We assume that for every sort $s \in S$, X_s is a given countably infinite set of symbols which we use for (the denotation of) variables of sort s.

Terms

Terms over Σ (resp., Σ-terms) are formed according to these syntactic rules:

$$
\begin{array}{llll}
t, u, v & ::= & x & , x \in X & \text{(variable)} \\
 & | & f(s_1, ..., s_n) & , f/n \in \Omega & \text{(functional term)}
\end{array}
$$

By $\mathrm{T}_\Sigma(X)$ we denote the set of Σ-terms (over X).

A term not containing any variable is called a ground term.

By T_Σ we denote the set of Σ-ground terms.

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By T_Σ we denote the set of Σ-ground terms.

Many-sorted case:

a variable $x \in X_s$ is a term of sort s

if $a(f) = s_1 \dots s_n \to s$, and t_i are terms of sort s_i, $i = 1, \dots, n$ then $f(t_1, ..., t_n)$ is a term of sort s.

Terms

In other words, terms are formal expressions with well-balanced brackets which we may also view as marked, ordered trees.

The markings are function symbols or variables.

The nodes correspond to the subterms of the term.

A node v that is marked with a function symbol f of arity n has exactly n subtrees representing the n immediate subterms of v.

Atoms

Atoms (also called atomic formulas) over Σ are formed according to this syntax:

$$
\begin{array}{llll}
A, B & ::= & p(t_1, \ldots, t_m) & , \; p/m \in \Pi \\
\Big[& \mid & (t \approx t') & \text{(equation)} \quad \Big]
\end{array}
$$

Whenever we admit equations as atomic formulas we are in the realm of first-order logic with equality. Admitting equality does not really increase the expressiveness of first-order logic, (cf. exercises). But deductive systems where equality is treated specifically can be much more efficient.

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Many-sorted case:

If $a(p) = s_1 \ldots s_m$, we require that t_i is a term of sort s_i for $i = 1, \ldots, m$.

Literals

$$L ::= A \quad \text{(positive literal)}$$
$$\mid \neg A \quad \text{(negative literal)}$$

Clauses

$$
\begin{array}{lcll}
C, D & ::= & \bot & \text{(empty clause)} \\
 & \mid & L_1 \vee \ldots \vee L_k, \;\; k \geq 1 & \text{(non-empty clause)}
\end{array}
$$

General First-Order Formulas

$\mathsf{F}_\Sigma(X)$ is the set of first-order formulas over Σ defined as follows:

$$
\begin{array}{rcll}
F, G, H & ::= & \bot & \text{(falsum)} \\
& \mid & \top & \text{(verum)} \\
& \mid & A & \text{(atomic formula)} \\
& \mid & \neg F & \text{(negation)} \\
& \mid & (F \wedge G) & \text{(conjunction)} \\
& \mid & (F \vee G) & \text{(disjunction)} \\
& \mid & (F \rightarrow G) & \text{(implication)} \\
& \mid & (F \leftrightarrow G) & \text{(equivalence)} \\
& \mid & \forall x F & \text{(universal quantification)} \\
& \mid & \exists x F & \text{(existential quantification)}
\end{array}
$$

Notational Conventions

We omit brackets according to the following rules:

- $\neg \quad >_p \quad \wedge \quad >_p \quad \vee \quad >_p \quad \rightarrow \quad >_p \quad \leftrightarrow$
  (binding precedences)
- $\vee$ and $\wedge$ are associative and commutative
- $\rightarrow$ is right-associative

$Qx_1, \dots, x_n\, F$ abbreviates $Qx_1 \dots Qx_n\, F$.

Notational Conventions

We use infix-, prefix-, postfix-, or mixfix-notation with the usual operator precedences.

Examples:

$$
\begin{array}{rcc}
s + t * u & \text{for} & +(s, *(t, u)) \\
s * u \leq t + v & \text{for} & \leq (*(s, u), +(t, v)) \\
-s & \text{for} & -(s) \\
0 & \text{for} & 0()
\end{array}
$$

Conventions

In what follows we will use the following conventions:

constants (0-ary function symbols) are denoted with $a, b, c, d, \ldots$

function symbols with arity ≥ 1 are denoted

- $f, g, h, \ldots$ if the formulae are interpreted into arbitrary algebras
- $+, -, s, \ldots$ if the intended interpretation is into numerical domains

predicate symbols with arity 0 are denoted $P, Q, R, S, \ldots$

predicate symbols with arity ≥ 1 are denoted

- $p, q, r, \ldots$ if the formulae are interpreted into arbitrary algebras
- $\leq, \geq, <, >$ if the intended interpretation is into numerical domains

variables are denoted $x, y, z, \ldots$

Example: Peano Arithmetic

Signature:

$$\Sigma_{PA} = (\Omega_{PA}, \ \Pi_{PA})$$
$$\Omega_{PA} = \{0/0, \ +/2, \ */2, \ s/1\}$$
$$\Pi_{PA} = \{\leq /2, \ < /2\}$$
$$+, *, <, \leq \text{ infix; } * \ >_p \ + \ >_p \ < \ >_p \ \leq$$

Examples of formulas over this signature are:

$$\forall x, y(x \leq y \leftrightarrow \exists z(x + z \approx y))$$
$$\exists x \forall y(x + y \approx y)$$
$$\forall x, y(x * s(y) \approx x * y + x)$$
$$\forall x, y(s(x) \approx s(y) \rightarrow x \approx y)$$
$$\forall x \exists y(x < y \wedge \neg \exists z(x < z \wedge z < y))$$

Remarks About the Example

We observe that the symbols $\leq$, $<$, 0, s are redundant as they can be defined in first-order logic with equality just with the help of $+$. The first formula defines $\leq$, while the second defines zero. The last formula, respectively, defines s.

Eliminating the existential quantifiers by Skolemization (cf. below) reintroduces the "redundant" symbols.

Consequently there is a *trade-off* between the complexity of the quantification structure and the complexity of the signature.

Example: Specifying LISP lists

Signature:

$$\Sigma_{\text{Lists}} = (\Omega_{\text{Lists}}, \Pi_{\text{Lists}})$$

$$\Omega_{\text{Lists}} = \{\text{car}/1, \text{cdr}/1, \text{cons}/2\}$$

$$\Pi_{\text{Lists}} = \emptyset$$

Examples of formulae:

$$\forall x, y \quad \text{car}(\text{cons}(x, y)) \approx x$$
$$\forall x, y \quad \text{cdr}(\text{cons}(x, y)) \approx y$$
$$\forall x \quad \text{cons}(\text{car}(x), \text{cdr}(x)) \approx x$$

Many-sorted signatures

Example:

Signature

$S = \{\text{array}, \text{index}, \text{element}\}$ set of sorts

$\Omega = \{\text{read}, \text{write}\}$

$$a(\text{read}) = \text{array} \times \text{index} \to \text{element}$$

$$a(\text{write}) = \text{array} \times \text{index} \times \text{element} \to \text{array}$$

$\Pi = \emptyset$

$X = \{X_s \mid s \in S\}$

Examples of formulae:

$\forall x : \text{array} \;\; \forall i : \text{index} \;\; \forall j : \text{index} \;\; (i \approx j \to \text{write}(x, i, \text{read}(x, j)) \approx x)$

$\forall x : \text{array} \; \forall y : \text{array} \;\; (x \approx y \leftrightarrow \forall i : \text{index} \;\; (\text{read}(x, i) \approx \text{read}(y, i)))$

Bound and Free Variables

In QxF, $Q \in \{\exists, \forall\}$, we call F the scope of the quantifier Qx.
An *occurrence* of a variable x is called bound, if it is inside the scope of a quantifier Qx.
Any other occurrence of a variable is called free.

Formulas without free variables are also called closed formulas or sentential forms.

Formulas without variables are called ground.

Bound and Free Variables

Example:

$$\forall\, y \quad (\overbrace{\forall\, x \quad \overbrace{p(x)}^{\textit{scope}} \quad \rightarrow \quad q(x, y)}^{\textit{scope}})$$

The occurrence of y is bound, as is the first occurrence of x. The second occurrence of x is a free occurrence.

Substitutions

Substitution is a fundamental operation on terms and formulas that occurs in all inference systems for first-order logic.

In general, substitutions are mappings

$$\sigma : X \to \mathsf{T}_\Sigma(X)$$

such that the domain of σ, that is, the set

$$dom(\sigma) = \{x \in X \mid \sigma(x) \neq x\},$$

is finite. The set of variables introduced by σ, that is, the set of variables occurring in one of the terms $\sigma(x)$, with $x \in dom(\sigma)$, is denoted by $codom(\sigma)$.

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Many-sorted case: Substitutions must be sort-preserving:
If x is a variable of sort s, then $\sigma(x)$ must be a term of sort s.

Substitutions

Substitutions are often written as $[s_1/x_1, \ldots, s_n/x_n]$, with x_i pairwise distinct, and then denote the mapping

$$[s_1/x_1, \ldots, s_n/x_n](y) = \begin{cases} s_i, & \text{if } y = x_i \\ y, & \text{otherwise} \end{cases}$$

We also write $x\sigma$ for $\sigma(x)$.

The modification of a substitution σ at x is defined as follows:

$$\sigma[x \mapsto t](y) = \begin{cases} t, & \text{if } y = x \\ \sigma(y), & \text{otherwise} \end{cases}$$

Why Substitution is Complicated

We define the application of a substitution σ to a term t or formula F by structural induction over the syntactic structure of t or F by the equations depicted on the next page.

In the presence of quantification it is surprisingly complex:
We need to make sure that the (free) variables in the codomain of σ are not *captured* upon placing them into the scope of a quantifier Qy, hence the bound variable must be renamed into a "fresh", that is, previously unused, variable z.

Application of a Substitution

"Homomorphic" extension of σ to terms and formulas:

$$
\begin{aligned}
f(s_1, \ldots, s_n)\sigma &= f(s_1\sigma, \ldots, s_n\sigma) \\
\bot\sigma &= \bot \\
\top\sigma &= \top \\
p(s_1, \ldots, s_n)\sigma &= p(s_1\sigma, \ldots, s_n\sigma) \\
(u \approx v)\sigma &= (u\sigma \approx v\sigma) \\
\neg F\sigma &= \neg(F\sigma) \\
(F \rho G)\sigma &= (F\sigma\, \rho\, G\sigma) \ ; \quad \text{for each binary connective } \rho \\
(Qx\, F)\sigma &= Qz\,(F\,[x \mapsto z]\sigma) \ ; \quad \text{with } z \text{ a fresh variable}
\end{aligned}
$$

2.2 Semantics

To give semantics to a logical system means to define a notion of truth for the formulas. The concept of truth that we will now define for first-order logic goes back to Tarski.

As in the propositional case, we use a two-valued logic with truth values "true" and "false" denoted by 1 and 0, respectively.

Structures

A Σ-algebra (also called Σ-interpretation or Σ-structure) is a triple

$$\mathcal{A} = (U,\ (f_{\mathcal{A}} : U^n \to U)_{f/n \in \Omega},\ (p_{\mathcal{A}} \subseteq U^m)_{p/m \in \Pi})$$

where $U \neq \emptyset$ is a set, called the universe of $\mathcal{A}$.

Normally, by abuse of notation, we will have $\mathcal{A}$ denote both the algebra and its universe.

By Σ-Alg we denote the class of all Σ-algebras.

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Normally, by abuse of notation, we will have $\mathcal{A}$ denote both the algebra and its universe.

By Σ-Alg we denote the class of all Σ-algebras.

A many-sorted Σ-algebra (also called Σ-interpretation or Σ-structure), where $\Sigma = (S, \Omega, \Pi)$ is a triple

$$\mathcal{A} = (\{U_s\}_{s \in S},\ (f_{\mathcal{A}} : U_{s_1} \times \ldots \times U_{s_n} \to U_s)_{\substack{f \in \Omega, \\ a(f) = s_1 \ldots s_n \to s}}\ (p_{\mathcal{A}} : U_{s_1} \times \ldots \times U_{s_m} \to \{0, 1\})_{\substack{p \in \Pi \\ a(p) = s_1 \ldots s_m}})$$

where $U_s \neq \emptyset$ is a set, called the universe of $\mathcal{A}$ of sort s.

Assignments

A variable has no intrinsic meaning. The meaning of a variable has to be defined externally (explicitly or implicitly in a given context) by an assignment.

A (variable) assignment, also called a valuation (over a given Σ-algebra $\mathcal{A}$), is a map $\beta : X \to \mathcal{A}$.

Variable assignments are the semantic counterparts of substitutions.

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Variable assignments are the semantic counterparts of substitutions.

Many-sorted case:

$\beta = \{\beta_s\}_{s \in S}, \beta_s : X_s \to U_s$