Decision Procedures for Verification

Combinations of Decision Procedures (2)

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Last time

Combinations of Decision Procedures

Combination of theories over disjoint signatures

The Nelson/Oppen procedure

Given: $\mathcal{T}_1, \mathcal{T}_2$ first-order theories with signatures Σ_1, Σ_2

Assume that $\Sigma_1 \cap \Sigma_2 = \emptyset$ (share only $\approx$)

P_i decision procedures for satisfiability of ground formulae w.r.t. $\mathcal{T}_i$

ϕ quantifier-free formula over $\Sigma_1 \cup \Sigma_2$

Task: Check whether ϕ is satisfiable w.r.t. $\mathcal{T}_1 \cup \mathcal{T}_2$

Note: Restrict to conjunctive quantifier-free formulae

$\phi \mapsto DNF(\phi)$

$DNF(\phi)$ satisfiable in $\mathcal{T}$ iff one of the disjuncts satisfiable in $\mathcal{T}$

The Nelson-Oppen algorithm

ϕ conjunction of literals

Step 1. Purification $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \phi \mapsto (\mathcal{T}_1 \cup \phi_1) \cup (\mathcal{T}_2 \cup \phi_2)$:

where ϕ_i is a pure Σ_i-formula and $\phi_1 \wedge \phi_2$ is equisatisfiable with ϕ.

Step 2. Propagation.

The decision procedure for ground satisfiability for $\mathcal{T}_1$ and $\mathcal{T}_2$ fairly exchange information concerning entailed unsatisfiability of constraints in the shared signature

i.e. clauses over the shared variables.

until an inconsistency is detected or a saturation state is reached.

The Nelson-Oppen algorithm

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where ϕ_i is a pure Σ_i-formula and $\phi_1 \wedge \phi_2$ is equisatisfiable with ϕ.

> not problematic; requires linear time

Step 2. Propagation.

The decision procedure for ground satisfiability for $\mathcal{T}_1$ and $\mathcal{T}_2$ fairly exchange information concerning entailed unsatisfiability of constraints in the shared signature i.e. clauses over the shared variables.

until an inconsistency is detected or a saturation state is reached.

> not problematic; termination guaranteed
> Sound: if inconsistency detected input unsatisfiable
> Complete: under additional assumptions

Implementation

ϕ conjunction of literals

Step 1. Purification: $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \phi \mapsto (\mathcal{T}_1 \cup \phi_1) \cup (\mathcal{T}_2 \cup \phi_2)$,
where ϕ_i is a pure Σ_i-formula and $\phi_1 \wedge \phi_2$ is equisatisfiable with ϕ.

Step 2. Propagation: The decision procedure for ground satisfiability for $\mathcal{T}_1$ and $\mathcal{T}_2$ fairly exchange information concerning entailed unsatisfiability of constraints in the shared signature
i.e. clauses over the shared variables.

until an inconsistency is detected or a saturation state is reached.

How to implement Propagation?

Guessing: guess a maximal set of literals containing the shared variables; check it for $\mathcal{T}_i \cup \phi_i$ consistency.

Backtracking: identify disjunction of equalities between shared variables entailed by $\mathcal{T}_i \cup \phi_i$; make case split by adding some of these equalities to ϕ_1, ϕ_2. Repeat as long as possible.

The Nelson-Oppen algorithm

Termination: only finitely many shared variables to be identified

The Nelson-Oppen algorithm

Termination: only finitely many shared variables to be identified

Soundness: If procedure answers "unsatisfiable" then ϕ is unsatisfiable

Proof: Assume that ϕ is satisfiable. Then $\phi_1 \wedge \phi_2$ satisfiable.

- The procedure cannot answer "unsatisfiable" in Step 2.
- Let $(\mathcal{M}, \beta) \models \phi_1 \wedge \phi_2$. Assume that $(\mathcal{M}, \beta) \models \bigwedge_{(c_i,c_j)\in E} c_i \approx c_j \wedge \bigwedge_{(c_i,c_j)\notin E} c_i \not\approx c_j$

Then
$$(\mathcal{M}_{|\Sigma_1}, \beta) \models \phi_1 \wedge \bigwedge_{(c_i,c_j)\in E} c_i \approx c_j$$
$$(\mathcal{M}_{|\Sigma_2}, \beta) \models \phi_2 \wedge \bigwedge_{(c_i,c_j)\in E} c_i \approx c_j$$

Guessing: $\bigwedge_{(c_i,c_j)\in E} c_i \approx c_j \wedge \bigwedge_{(c_i,c_j)\notin E} c_i \not\approx c_j$ "satisfiable arrangement".

Backtracking: Procedure answers satisfiable on the corresponding branch.

The Nelson-Oppen algorithm

Termination: only finitely many shared variables to be identified

Soundness: If procedure answers "unsatisfiable" then ϕ is unsatisfiable

Completeness: Under additional hypotheses

Completeness

Example:

E_1	E_2
$f(g(x), g(y)) \approx x$	$k(x) \approx k(x)$
$f(g(x), h(y)) \approx y$	
non-trivial	non-trivial

$g(c) \approx h(c) \land k(c) \not\approx c$

$g(c) \approx h(c)$	$k(c) \not\approx c$
satisfiable in E_1	satisfiable in E_2

no equations between shared variables; Nelson-Oppen answers "satisfiable"

Completeness

Example:

E_1	E_2
$f(g(x), g(y)) \approx x$	$k(x) \approx k(x)$
$f(g(x), h(y)) \approx y$	
non-trivial	non-trivial

$g(c){\approx}h(c) \wedge k(c){\not\approx}c$

$g(c){\approx}h(c)$	$k(c){\not\approx}c$
satisfiable in E_1	satisfiable in E_2

no equations between shared variables; Nelson-Oppen answers "satisfiable"

A model of E_1 satisfies $g(c) \approx h(c)$ iff $\exists e \in A$ s.t. $g(e) = h(e)$.

Then, for all $a \in A$: $a = f_A(g(a), g(e)) = f_A(g(a), h(e)) = e$

$g(c){\approx}h(c) \wedge k(c){\not\approx}c$ unsatisfiable

Completeness

Another example

$\mathcal{T}_1$ theory admitting models of cardinality at most 2

$\mathcal{T}_2$ theory admitting models of any cardinality

$$f_1 \in \Sigma_1, f_2 \in \Sigma_2 \quad \text{such that} \quad \mathcal{T}_i \not\models \forall x, y \quad f_i(x) = f_i(y).$$

$$\phi \;=\; f_1(c_1) \not\approx f_1(c_2) \quad \wedge \quad f_2(c_1) \not\approx f_2(c_3) \quad \wedge \quad f_2(c_2) \not\approx f_2(c_3)$$

$$\phi_1 \;=\; f_1(c_1) \not\approx f_1(c_2) \qquad \phi_2 \;=\; f_2(c_1) \not\approx f_2(c_3) \;\wedge\; f_2(c_2) \not\approx f_2(c_3)$$

The Nelson-Oppen procedure returns "satisfiable"

$$\mathcal{T}_1 \cup \mathcal{T}_2 \models \forall x, y, z (f_1(x) \not\approx f_1(y) \;\wedge\; f_2(x) \not\approx f_2(z) \;\wedge\; f_2(y) \not\approx f_2(z) \\ \rightarrow (x \not\approx y \wedge x \not\approx z \wedge y \not\approx z))$$

$f_1(c_1) \not\approx f_1(c_2) \quad \wedge \quad f_2(c_1) \not\approx f_2(c_3) \quad \wedge \quad f_2(c_2) \not\approx f_2(c_3)$ unsatisfiable

Completeness

Cause of incompleteness

There exist formulae satisfiable in finite models of bounded cardinality

Solution: Consider stably infinite theories.

$\mathcal{T}$ is stably infinite iff for every quantifier-free formula ϕ
ϕ satisfiable in $\mathcal{T}$ iff ϕ satisfiable in an infinite model of $\mathcal{T}$.

Note: This restriction is not mentioned in [Nelson Oppen 1979];

introduced by Oppen in 1980.

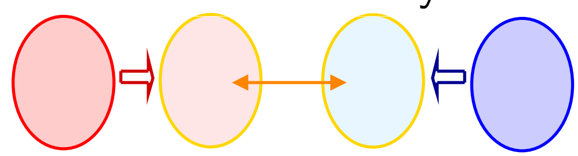
Completeness

Guessing version: C set of constants shared by ϕ_1, ϕ_2

R equiv. relation assoc. with partition of $C \mapsto ar(C, R) = \bigwedge_{R(c,d)} c \approx d \wedge \bigwedge_{\neg R(c,d)} c \not\approx d$

Lemma. Assume that there exists a partition of C s.t. $\phi_i \wedge ar(C, R)$ is $\mathcal{T}_i$-satisfiable. Then $\phi_1 \wedge \phi_2$ is $\mathcal{T}_1 \cup \mathcal{T}_2$-satisfiable.

Idea of proof: Let $\mathcal{A}_i \in \text{Mod}(\mathcal{T}_i)$ s.t. $\mathcal{A}_i \models \phi_i \wedge ar(C, R)$. Then $c_{A_1} = d_{A_1}$ iff $c_{A_2} = d_{A_2}$.
Let $i : \{c_{A_1} \mid c \in C\} \to \{c_{A_2} \mid c \in C\}$, $i(c_{A_1}) = c_{A_2}$ well-defined; bijection.
Stable infinity: can assume w.l.o.g. that $\mathcal{A}_1, \mathcal{A}_2$ have the same cardinality
Let $h : \mathcal{A}_1 \to \mathcal{A}_2$ bijection s.t. $h(c_{A_1}) = c_{A_2}$



Use h to transfer the Σ_1-structure on $\mathcal{A}_2$.

Theorem. If $\mathcal{T}_1, \mathcal{T}_2$ are both stably infinite and the shared signature is empty then the Nelson-Oppen procedure is sound, complete and terminating. Thus, it transfers decidability of ground satisfiability from $\mathcal{T}_1, \mathcal{T}_2$ to $\mathcal{T}_1 \cup \mathcal{T}_2$.

Complexity

Main sources of complexity:

(i) transformation of the formula in DNF

(ii) propagation

(a) decide whether there is a disjunction of equalities between variables

(b) investigate different branches corresponding to disjunctions

Complexity

Main sources of complexity:

(i) transformation of the formula in DNF

(ii) propagation

$\mathcal{T}$ is convex	iff	for every quantifier-free conjunctive formula ϕ,
		$\phi \models \bigvee_i x_i \approx y_i$ implies $\phi \models x_j \approx y_j$ for some j.

$\mapsto$ No branching

Complexity

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$\mapsto$ No branching

Examples of convex theories:

- The theory of uninterpreted function symbols
- $LI(\mathbb{Q})$

Examples of theories which are not convex:

- $LI(\mathbb{Z})$

Complexity

Theorem. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be convex and stably infinite; $\Sigma_1 \cap \Sigma_2 = \emptyset$

If satisfiability of conjunctions of literals in $\mathcal{T}_i$ is in PTIME

Then satisfiability of conjunctions of literals in $\mathcal{T}_1 \cup \mathcal{T}_2$ is in PTIME

In general: non-deterministic procedure

Theorem. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be convex and stably infinite; $\Sigma_1 \cap \Sigma_2 = \emptyset$

If satisfiability of conjunctions of literals in $\mathcal{T}_i$ is in NP

Then satisfiability of conjunctions of literals in $\mathcal{T}_1 \cup \mathcal{T}_2$ is in NP

From conjunctions to arbitrary combinations

Until now:

check satisfiability for conjunctions of literals

Question:

how to check satisfiability of sets of clauses?

Overview

- Propositional logic
  - resolution
  - DPLL
- First-order logic
  - resolution

Satisfiability w.r.t. theories

- Ground formulae
  - conjunctions of literals:
    specialized methods
  - clauses: DPLL(T) $\Leftarrow$ TODAY
- Formulae with quantifiers
  - reduction to SAT for ground formulae
    instantiation $\Leftarrow$ NEXT WEEK
    (situations when sound and complete)
  - resolution (mod T)

3.6 The $\mathit{DPLL}(\mathcal{T})$ algorithm

Reminder: Propositional SAT

The DPLL algorithm

A succinct formulation

State: $M||F$,

where:

- M partial assignment (sequence of literals),

  some literals are annotated (L^d: decision literal)

- F clause set.

A succinct formulation

UnitPropagation

$M||F, C \vee L \Rightarrow M, L||F, C \vee L$ if $M \models \neg C$, and L undef. in M

Decide

$M||F \Rightarrow M, L^d||F$ if L or $\neg L$ occurs in F, L undef. in M

Fail

$M||F, C \Rightarrow \text{Fail}$ if $M \models \neg C$, M contains no decision literals

Backjump

$$M, L^d, N||F \Rightarrow M, L'||F \quad \text{if} \begin{cases} \text{there is some clause } C \vee L' \text{ s.t.:} \\ F \models C \vee L', M \models \neg C, \\ L' \text{ undefined in } M \\ L' \text{ or } \neg L' \text{ occurs in } F. \end{cases}$$

Example

Assignment:	Clause set:	
$\emptyset$	$\|\|\neg P_1 \lor P_2, \neg P_3 \lor P_4, \neg P_5 \lor \neg P_6, P_6 \lor \neg P_5 \lor \neg P_2$	$\Rightarrow$ (Decide)
P_1^d	$\|\|\neg P_1 \lor P_2, \neg P_3 \lor P_4, \neg P_5 \lor \neg P_6, P_6 \lor \neg P_5 \lor \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2$	$\|\|\neg P_1 \lor P_2, \neg P_3 \lor P_4, \neg P_5 \lor \neg P_6, P_6 \lor \neg P_5 \lor \neg P_2$	$\Rightarrow$ (Decide)
$P_1^d P_2 P_3^d$	$\|\|\neg P_1 \lor P_2, \neg P_3 \lor P_4, \neg P_5 \lor \neg P_6, P_6 \lor \neg P_5 \lor \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2 P_3^d P_4$	$\|\|\neg P_1 \lor P_2, \neg P_3 \lor P_4, \neg P_5 \lor \neg P_6, P_6 \lor \neg P_5 \lor \neg P_2$	$\Rightarrow$ (Decide)
$P_1^d P_2 P_3^d P_4 P_5^d$	$\|\|\neg P_1 \lor P_2, \neg P_3 \lor P_4, \neg P_5 \lor \neg P_6, P_6 \lor \neg P_5 \lor \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2 P_3^d P_4 P_5^d \neg P_6$	$\|\|\neg P_1 \lor P_2, \neg P_3 \lor P_4, \neg P_5 \lor \neg P_6, P_6 \lor \neg P_5 \lor \neg P_2$	$\Rightarrow$ (Backtrack)
$P_1^d P_2 P_3^d P_4 \neg P_5$	$\|\|\neg P_1 \lor P_2, \neg P_3 \lor P_4, \neg P_5 \lor \neg P_6, P_6 \lor \neg P_5 \lor \neg P_2$	

DPLL with learning

The DPLL system with learning consists of the four transition rules of the Basic DPLL system, plus the following two additional rules:

Learn

$M||F \Rightarrow M||F, C$ if all atoms of C occur in F and $F \models C$

Forget

$M||F, C \Rightarrow M||F$ if $F \models C$

In these two rules, the clause C is said to be learned and forgotten, respectively.

SAT Modulo Theories (SMT)

Some problems are more naturally expressed in richer logics than just propositional logic, e.g:

- Software/Hardware verification needs reasoning about equality, arithmetic, data structures, ...

SMT consists of deciding the satisfiability of a ground 1st-order formula with respect to a background theory T

Example 1: $\mathcal{T}$ is Equality with Uninterpreted Functions (UIF):

$$f(g(a)) \not\approx f(c) \;\vee\; g(a) \approx d, \quad g(a) \approx c, \quad c \not\approx d$$

Example 2: for combined theories:

$$A \approx \mathrm{write}(B, a+1, 4), \quad \mathrm{read}(A, b+3) \approx 2 \;\vee\; f(a-1) \not\approx f(b+1)$$

SAT Modulo Theories (SMT)

The “very eager” approach to SMT

Method:

– translate problem into equisatisfiable propositional formula;

– use off-the-shelf SAT solver

- Why “eager”?

  Search uses all theory information from the beginning

- Characteristics:

  + Can use best available SAT solver

  – Sophisticated encodings are needed for each theory

  – Sometimes translation and/or solving too slow

Main Challenge for alternative approaches is to combine:

- DPLL-based techniques for handling the boolean structure

- Efficient theory solvers for conjunctions of $\mathcal{T}$-literals

SAT Modulo Theories (SMT)

"Lazy" approaches to SMT: Idea

Example: consider $\mathcal{T} = UIF$ and the following set of clauses:

$$\underbrace{f(g(a)) \not\approx f(c)}_{\neg P_1} \vee \underbrace{g(a) \approx d}_{P_2}, \quad \underbrace{g(a) \approx c}_{P_3}, \quad \underbrace{c \not\approx d}_{\neg P_4}$$

1. Send $\{\neg P_1 \vee P_2,\ P_3,\ \neg P_4\}$ to SAT solver

   SAT solver returns model $[\neg P_1, P_3, \neg P_4]$
   Theory solver says $\neg P_1 \wedge P_3 \wedge \neg P_4$ is $\mathcal{T}$-inconsistent

2. Send $\{\neg P_1 \vee P_2,\ P_3,\ \neg P_4,\ P_1 \vee \neg P_3 \vee P_4\}$ to SAT solver

   SAT solver returns model $[P_1, P_2, P_3, \neg P_4]$
   Theory solver says $P_1 \wedge P_2 \wedge P_3 \wedge \neg P_4$ is $\mathcal{T}$-inconsistent

3. Send $\{\neg P_1 \vee P_2, P_3, \neg P_4, P_1 \vee \neg P_3 \vee P_4, \neg P_1 \vee \neg P_2 \vee \neg P_3 \vee P_4\}$ to SAT solver

   SAT solver says UNSAT

SAT Modulo Theories (SMT)

Optimized lazy approach

LA	• Check T-consistency only of full propositional models
OLA	• Check T-consistency of partial assignment while being built
LA	• Given a T-inconsistent assignment M, add $\neg M$ as a clause
OLA	• Given a T-inconsistent assignment M, find an explanation (a small T-inconsistent subset of M) and add it as a clause
LA	• Upon a T-inconsistency, add clause and restart
OLA	• Upon a T-inconsistency, do conflict analysis of the explanation and Backjump

SAT Modulo Theories (SMT)

"Lazy" approaches to SMT

- Why "lazy"?

Theory information used only lazily, when checking $\mathcal{T}$-consistency of propositional models

- Characteristics:

  + Modular and flexible
  – Theory information does not guide the search
    (only validates a posteriori)

Tools: CVC-Lite, ICS, MathSAT, TSAT++, Verifun, ...

“Lazy” approaches to SMT

Lazy theory learning:

$$M, L, M_1 || F \Rightarrow \emptyset || F, \neg L_1 \vee \cdots \vee \neg L_n \vee \neg L \quad \text{if} \begin{cases} M, L, M_1 \models F \\ \{L_1, \ldots, L_n\} \subseteq M \\ L_1 \wedge \cdots \wedge L_n \wedge L \models_{\mathcal{T}} \perp \end{cases}$$

Lazy theory learning + no repetitions

$$M, L, M_1 || F \Rightarrow \emptyset || F, \neg L_1 \vee \cdots \vee \neg L_n \vee \neg L \quad \text{if} \begin{cases} \{L_1, \ldots, L_n\} \subseteq M \\ L_1 \wedge \cdots \wedge L_n \wedge L \models_{\mathcal{T}} \perp \\ \neg L_1 \vee \cdots \vee \neg L_n \vee \neg L \notin F \end{cases}$$

DPLL(T) Rules

UnitPropagation

$M||F, C \vee L \Rightarrow M, L||F, C \vee L$ if $M \models \neg C$, and L undef. in M

Decide

$M||F \Rightarrow M, L^d||F$ if L occurs in F, L undef. in M

Fail

$M||F, C \Rightarrow \text{Fail}$ if $M \models \neg C$, no backtrack possible

Backjump

$$M, L^d, N||F \Rightarrow M, L'||F \quad \text{if} \begin{cases} \text{there is some clause } C \vee L' \text{ s.t.:} \\ F \models C \vee L', M \models \neg C, \\ L' \text{ undefined in } M \\ L' \text{ or } \neg L' \text{ occurs in } F. \end{cases}$$

Restart/Learn

$M||F \Rightarrow \emptyset||F, F'$ if $F \models F'$, F' obtained from M, F

TPropagation

$M||F \Rightarrow M, L||F$ if $M \models_{\mathcal{T}} L$

DPLL(T) Example

Consider again same example with UIF:

$$\underbrace{f(g(a)) \not\approx f(c)}_{\neg P_1} \vee \underbrace{g(a) \approx d}_{P_2}, \quad \underbrace{g(a) \approx c}_{P_3}, \quad \underbrace{c \not\approx d}_{\neg P_4}$$

$$\begin{array}{lll}
\emptyset & \| \neg P_1 \vee P_2, P_3, \neg P_4 & \Rightarrow (\mathit{UnitPropagation}) \\
P_3 & \| \neg P_1 \vee P_2, P_3, \neg P_4 & \Rightarrow (\mathit{TPropagation}) \\
P_3 P_1 & \| \neg P_1 \vee P_2, P_3, \neg P_4 & \Rightarrow (\mathit{UnitPropagation}) \\
P_3 P_1 P_2 & \| \neg P_1 \vee P_2, P_3, \neg P_4 & \Rightarrow (\mathit{TPropagation}) \\
P_3 P_1 P_2 P_4 & \| \neg P_1 \vee P_2, P_3, \neg P_4 & \Rightarrow \mathit{fail}
\end{array}$$

No search in this example

Termination

Idea: $DPLL(T)$ terminates if no clause is learned infinitely many times, since only finitely many such new clauses (built over input literals) exist.

Theorem. There exists no infinite sequence of the form

$$\emptyset || F \Rightarrow S_1 \Rightarrow S_2 ...$$

if no clause C is learned by Reset & Learn/Lazy Theory Learning infinitely many times along a sequence.

A similar termination result holds also for the DPLL(T) approach with Theory Propagation.

Termination

Theorem. There exist no infinite sequences of the form $\emptyset || F \Rightarrow S_1 \Rightarrow S_2 ...$

Proof. (Idea) We define a well-founded strict partial ordering $\succ$ on states, and show that each rule application $M||F \Rightarrow M'||F'$ is decreasing with respect to this ordering, i.e., $M||F \succ M'||F'$.

Let M be of the form $M_0, L_1, M_1, ...L_p, M_p$, where $L_1, ..., L_p$ are all the decision literals of M. Similarly, let M' be $M'_0, L'_1, M'_1, ...L'_{p'}, M'_{p'}$.

Let N be the number of distinct atoms (propositional variables) in F.

(Note that p, p' and the length of M and M' are always smaller than or equal to N.)

Termination

Theorem. There exist no infinite sequences of the form $\emptyset || F \Rightarrow S1 \Rightarrow \ldots$

Proof. (continued)
Let $m(M)$ be $N - \text{length}(M)$ (nr. of literals missing in M for M to be total).

Define: $M_0 L_1 M_1 \ldots L_p M_p || F \succ M_0' L_1' M_1' \ldots L_{p'}' M_{p'}' || F'$ if

(i) there is some i with $0 \leq i \leq p, p'$ such that

$m(M_0) = m(M_0'), \ldots m(M_{i-1}) = m(M_{i-1}'), m(M_i) > m(M_i')$ or

(ii) $m(M_0) = m(M_0'), \ldots m(M_p) = m(M_p')$ and $m(M) > m(M')$.

Comparing the number of missing literals in sequences is a strict ordering (irreflexive and transitive) and it is well-founded, and hence this also holds for its lexicographic extension on tuples of sequences of bounded length.

No learning/forgetting: It is easy to see that all Basic DPLL rule applications are decreasing with respect to $\succ$ if fail is added as an additional minimal element. (The rules UnitPropagate and Backjump decrease by case (i) of the definition and Decide decreases by case (ii).)

Termination

Theorem. There exist no infinite sequences of the form $\emptyset || F \Rightarrow S1 \Rightarrow \ldots$

Note: Combine learning with basic DPLL(T): no clause learned infinitely many times.

Forget: For this termination condition to be fulfilled, applying at least one rule of the Basic DPLL system between any two Learn applications does not suffice. It suffices if, in addition, no clause generated with Learning is ever forgotten.

Soundness, Correctness, Termination

Lemma. If $\emptyset || F \Rightarrow^* M || F'$ then:

(1) All atoms in M and all atoms in F' are atoms of F.

(2) M: no literal more than once, no complementary literals

(3) F' is logically equivalent to F

(4) if $M = M_0 L_1 M_1 \ldots L_n M_n$ where L_i all decision literals
then $F, L_1, \ldots, L_i \models M_i$.

Lemma. If $\emptyset || F \Rightarrow^* M || F'$, where $M || F'$ is a final state wrt the Basic DPLL system and Lazy Theory Learning, then:

(1) All literals of F' are defined in M

(2) There is no clause C in F' such that $M \models \neg C$

(3) M is a model of F.

Soundness, Correctness, Termination

Lemma. If $\emptyset || F \Rightarrow^* M || F'$, where $M || F'$ is a final state wrt the Basic DPLL system and Lazy Theory Learning, then M is a $\mathcal{T}$-model of F.

Theorem. The Lazy Theory learning DPLL system provides a decision procedure for the satisfiability in $\mathcal{T}$ of CNF formulae F, that is:

1. $\emptyset || F \Rightarrow^*$ *fail* if, and only if, F is unsatisfiable in $\mathcal{T}$.
2. $\emptyset || F \Rightarrow^* M || F'$, where $M || F'$ is a final state wrt the Basic DPLL system and Lazy Theory Learning, if, and only if, F is satisfiable in $\mathcal{T}$.

Proof

(1) If $\emptyset || F \Rightarrow^*$ *fail* then there exists state $M || F'$ with $\emptyset || F \Rightarrow^* M || F' \Rightarrow$ *fail*, there is no decision literal in M and $M \models \neg C$ for some clause C in F. By the construction of M, $F \models M$, so $F \models \neg C$. Thus F is unsatisfiable.

To prove the converse, if $\emptyset || F \not\Rightarrow^*$ *fail* then by there must be a state $M || F'$ such that $\emptyset || F \Rightarrow^* M || F'$. Then $M \models F$, so F is satisfiable.

Soundness, Correctness, Termination

Lemma. If $\emptyset || F \Rightarrow^* M || F'$, where $M || F'$ is a final state wrt the Basic DPLL system and Lazy Theory Learning, then M is a $\mathcal{T}$-model of F.

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2. $\emptyset || F \Rightarrow^* M || F'$, where $M || F'$ is a final state wrt the Basic DPLL system and Lazy Theory Learning, if, and only if, F is satisfiable in $\mathcal{T}$.

Proof

2. If $\emptyset || F \Rightarrow^* M || F$ then F is satisfiable. Conversely, if $\emptyset || F \not\Rightarrow^* M || F$ then $\emptyset || F \Rightarrow^*$ *fail*, so F is unsatisfiable.

Termination, Soundness and Completeness

DPLL($\mathcal{T}$) with (eager) theory propagation

Lemma. If $\emptyset || F \Rightarrow M || F$ then M is $\mathcal{T}$-consistent.

Proof. This property is true initially, and all rules preserve it, by the fact that $M \models_{\mathcal{T}} L$ if, and only if, $M \cup \neg L$ is $\mathcal{T}$-inconsistent: the rules only add literals to M that are undefined in M, and Theory Propagate adds all literals L of F that are theory consequences of M, before any literal $\neg L$ making it $\mathcal{T}$-inconsistent can be added to M by any of the other rules.

Termination, Soundness and Completeness

DPLL($\mathcal{T}$) with (eager) theory propagation

Definition. A DPLL($\mathcal{T}$) procedure with Eager Theory Propagation for $\mathcal{T}$ is any procedure taking an input CNF F and computing a sequence $\emptyset || F \Rightarrow^* S$ where S is a final state wrt. Theory Propagate and the Basic DPLL system.

Theorem The DPLL system with eager theory propagation provides a decision procedure for the satisfiability in $\mathcal{T}$ of CNF formulae F, that is:

1. $\emptyset || F \Rightarrow^* \textit{fail}$ if, and only if, F is unsatisfiable in $\mathcal{T}$.
2. $\emptyset || F \Rightarrow^* M || F'$, where $M || F'$ is a final state wrt the Basic DPLL system and Theory Propagate, if, and only if, F is satisfiable in $\mathcal{T}$.
3. If $\emptyset || F \Rightarrow M || F'$, where $M || F'$ is a final state wrt the Basic DPLL system and Theory Propagate, then M is a $\mathcal{T}$-model of F.

Literature

Full proofs and further details can be found in:

Robert Nieuwenhuis, Albert Oliveras and Cesare Tinelli:
"Solving SAT and SAT Modulo Theories: From an Abstract Davis-Putnam-Logemann-Loveland Procedure to DPLL(T)"

Journal of the ACM, Vol. 53, No. 6, November 2006, pp.937-977.

SMT tools

SAT problems

Given: conjunction ϕ of prop. clauses
Task: check if ϕ satisfiable

Method: DPLL

- deterministic choices first
  unit resolution
  pure literal assignment
- case distinction (splitting)
- heuristics
  selection criteria for splitting
  backtracking
  conflict-driven learning

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SMT problems

Given: conjunction ϕ of clauses
Task: check if $\phi \models_{\mathcal{T}} \perp$

Method: DPLL($\mathcal{T}$)

- Boolean assignment found
  using DPLL
- ... and checked for $\mathcal{T}$-satisfiability
- the assignment can be partial
  and checked before splitting
- usual heuristics are used:
  non-chronological backtracking
  learning

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Systems implementing such specialized satisfiability problems: Yices, Barcelogic Tools, CVC lite,haRVey,Math-SAT,Z3,...are called (S)atisfiability (M)odulo (T)heory solvers.