Decision Procedures for Verification

First-Order Logic (2)

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Conventions

In what follows we will use the following conventions:

constants (0-ary function symbols) are denoted with $a, b, c, d, \ldots$

function symbols with arity ≥ 1 are denoted

- $f, g, h, \ldots$ if the formulae are interpreted into arbitrary algebras
- $+, -, s, \ldots$ if the intended interpretation is into numerical domains

predicate symbols with arity 0 are denoted $P, Q, R, S, \ldots$

predicate symbols with arity ≥ 1 are denoted

- $p, q, r, \ldots$ if the formulae are interpreted into arbitrary algebras
- $\leq, \geq, <, >$ if the intended interpretation is into numerical domains

variables are denoted $x, y, z, \ldots$

Until now:

Syntax (one-sorted signatures vs. many-sorted signatures)

Semantics Σ-structures and valuations

Structures

A Σ-algebra (also called Σ-interpretation or Σ-structure) is a triple

$$\mathcal{A} = (U,\ (f_{\mathcal{A}} : U^n \to U)_{f/n \in \Omega},\ (p_{\mathcal{A}} \subseteq U^m)_{p/m \in \Pi})$$

where $U \neq \emptyset$ is a set, called the universe of $\mathcal{A}$.

Normally, by abuse of notation, we will have $\mathcal{A}$ denote both the algebra and its universe.

By Σ-Alg we denote the class of all Σ-algebras.

A many-sorted Σ-algebra (also called Σ-interpretation or Σ-structure), where $\Sigma = (S, \Omega, \Pi)$ is a triple

$$\mathcal{A} = (\{U_s\}_{s \in S},\ (f_{\mathcal{A}} : U_{s_1} \times \ldots \times U_{s_n} \to U_s)_{\substack{f \in \Omega, \\ a(f) = s_1 \ldots s_n \to s}}\ (p_{\mathcal{A}} : U_{s_1} \times \ldots \times U_{s_m} \to \{0,1\})_{\substack{p \in \Pi \\ a(p) = s_1 \ldots s_m}})$$

where $U_s \neq \emptyset$ is a set, called the universe of $\mathcal{A}$ of sort s.

Assignments

A variable has no intrinsic meaning. The meaning of a variable has to be defined externally (explicitly or implicitly in a given context) by an assignment.

A (variable) assignment, also called a valuation (over a given Σ-algebra $\mathcal{A}$), is a map $\beta : X \to \mathcal{A}$.

Variable assignments are the semantic counterparts of substitutions.

Many-sorted case:

$\beta = \{\beta_s\}_{s \in S}, \beta_s : X_s \to U_s$

Value of a Term in $\mathcal{A}$ with Respect to β

By structural induction we define

$$\mathcal{A}(\beta) : \mathsf{T}_\Sigma(X) \to \mathcal{A}$$

as follows:

$$\mathcal{A}(\beta)(x) = \beta(x), \qquad x \in X$$

$$\mathcal{A}(\beta)(f(s_1, \ldots, s_n)) = f_{\mathcal{A}}(\mathcal{A}(\beta)(s_1), \ldots, \mathcal{A}(\beta)(s_n)), \qquad f/n \in \Omega$$

Value of a Term in $\mathcal{A}$ with Respect to β

In the scope of a quantifier we need to evaluate terms with respect to modified assignments. To that end, let $\beta[x \mapsto a] : X \to \mathcal{A}$, for $x \in X$ and $a \in \mathcal{A}$, denote the assignment

$$\beta[x \mapsto a](y) := \begin{cases} a & \text{if } x = y \\ \beta(y) & \text{otherwise} \end{cases}$$

Truth Value of a Formula in $\mathcal{A}$ with Respect to β

$\mathcal{A}(\beta) : \mathrm{F}_\Sigma(X) \to \{0,1\}$ is defined inductively as follows:

$$
\begin{aligned}
\mathcal{A}(\beta)(\bot) &= 0 \\
\mathcal{A}(\beta)(\top) &= 1 \\
\mathcal{A}(\beta)(p(s_1,\ldots,s_n)) = 1 &\quad\Leftrightarrow\quad (\mathcal{A}(\beta)(s_1),\ldots,\mathcal{A}(\beta)(s_n)) \in p_{\mathcal{A}} \\
\mathcal{A}(\beta)(s \approx t) = 1 &\quad\Leftrightarrow\quad \mathcal{A}(\beta)(s) = \mathcal{A}(\beta)(t) \\
\mathcal{A}(\beta)(\neg F) = 1 &\quad\Leftrightarrow\quad \mathcal{A}(\beta)(F) = 0 \\
\mathcal{A}(\beta)(F \rho G) &= \mathrm{B}_\rho(\mathcal{A}(\beta)(F), \mathcal{A}(\beta)(G)) \\
&\quad \text{with } \mathrm{B}_\rho \text{ the Boolean function associated with } \rho \\
\mathcal{A}(\beta)(\forall x F) &= \min_{a \in U}\{\mathcal{A}(\beta[x \mapsto a])(F)\} \\
\mathcal{A}(\beta)(\exists x F) &= \max_{a \in U}\{\mathcal{A}(\beta[x \mapsto a])(F)\}
\end{aligned}
$$

Example

The "Standard" Interpretation for Peano Arithmetic:

$$
\begin{array}{rcl}
U_{\mathbb{N}} & = & \{0, 1, 2, \ldots\} \\
0_{\mathbb{N}} & = & 0 \\
s_{\mathbb{N}} & : & n \mapsto n + 1 \\
+_{\mathbb{N}} & : & (n, m) \mapsto n + m \\
*_{\mathbb{N}} & : & (n, m) \mapsto n * m \\
\leq_{\mathbb{N}} & = & \{(n, m) \mid n \text{ less than or equal to } m\} \\
<_{\mathbb{N}} & = & \{(n, m) \mid n \text{ less than } m\}
\end{array}
$$

Note that $\mathbb{N}$ is just one out of many possible Σ_{PA}-interpretations.

Example

Values over $\mathbb{N}$ for Sample Terms and Formulas:

Under the assignment $\beta : x \mapsto 1, y \mapsto 3$ we obtain

$$\begin{aligned}
\mathbb{N}(\beta)(s(x) + s(0)) &= 3 \\
\mathbb{N}(\beta)(x + y \approx s(y)) &= 1 \\
\mathbb{N}(\beta)(\forall x, y(x + y \approx y + x)) &= 1 \\
\mathbb{N}(\beta)(\forall z\ z \leq y) &= 0 \\
\mathbb{N}(\beta)(\forall x \exists y\ x < y) &= 1
\end{aligned}$$

2.3 Models, Validity, and Satisfiability

F is valid in $\mathcal{A}$ under assignment β:

$$\mathcal{A}, \beta \models F \quad :\Leftrightarrow \quad \mathcal{A}(\beta)(F) = 1$$

F is valid in $\mathcal{A}$ ($\mathcal{A}$ is a model of F):

$$\mathcal{A} \models F \quad :\Leftrightarrow \quad \mathcal{A}, \beta \models F, \text{ for all } \beta \in X \to U_{\mathcal{A}}$$

F is valid (or is a tautology):

$$\models F \quad :\Leftrightarrow \quad \mathcal{A} \models F, \text{ for all } \mathcal{A} \in \Sigma\text{-alg}$$

F is called satisfiable iff there exist $\mathcal{A}$ and β such that $\mathcal{A}, \beta \models F$.
Otherwise F is called unsatisfiable.

Substitution Lemma

The following propositions, to be proved by structural induction, hold for all Σ-algebras $\mathcal{A}$, assignments β, and substitutions σ.

Lemma 2.3:

For any Σ-term t

$$\mathcal{A}(\beta)(t\sigma) = \mathcal{A}(\beta \circ \sigma)(t),$$

where $\beta \circ \sigma : X \to \mathcal{A}$ is the assignment $\beta \circ \sigma(x) = \mathcal{A}(\beta)(x\sigma)$.

Proposition 2.4:

For any Σ-formula F, $\mathcal{A}(\beta)(F\sigma) = \mathcal{A}(\beta \circ \sigma)(F)$.

Substitution Lemma

Corollary 2.5:

$\mathcal{A}, \beta \models F\sigma \quad \Leftrightarrow \quad \mathcal{A}, \beta \circ \sigma \models F$

These theorems basically express that the syntactic concept of substitution corresponds to the semantic concept of an assignment.

Entailment and Equivalence

F entails (implies) G (or G is a consequence of F), written $F \models G$

$:\Leftrightarrow$ for all $\mathcal{A} \in \Sigma\text{-alg}$ and $\beta \in X \to U_{\mathcal{A}}$,
whenever $\mathcal{A}, \beta \models F$ then $\mathcal{A}, \beta \models G$.

F and G are called equivalent

$:\Leftrightarrow$ for all $\mathcal{A} \in \Sigma\text{-alg}$ und $\beta \in X \to U_{\mathcal{A}}$ we have
$\mathcal{A}, \beta \models F \quad \Leftrightarrow \quad \mathcal{A}, \beta \models G$.

Entailment and Equivalence

Proposition 2.6:

F entails G iff $(F \to G)$ is valid

Proposition 2.7:

F and G are equivalent iff $(F \leftrightarrow G)$ is valid.

Extension to sets of formulas N in the "natural way", e.g., $N \models F$

$:\Leftrightarrow$ for all $\mathcal{A} \in \Sigma\text{-alg}$ and $\beta \in X \to U_{\mathcal{A}}$:
if $\mathcal{A}, \beta \models G$, for all $G \in N$, then $\mathcal{A}, \beta \models F$.

Validity vs. Unsatisfiability

Validity and unsatisfiability are just two sides of the same medal as explained by the following proposition.

Proposition 2.8:

$$F \text{ valid} \quad \Leftrightarrow \quad \neg F \text{ unsatisfiable}$$

Hence in order to design a theorem prover (validity checker) it is sufficient to design a checker for unsatisfiability.

Q: In a similar way, entailment $N \models F$ can be reduced to unsatisfiability. How?

Theory of a Structure

Let $\mathcal{A} \in \Sigma$-alg. The (first-order) theory of $\mathcal{A}$ is defined as

$$Th(\mathcal{A}) = \{G \in \mathsf{F}_\Sigma(X) \mid \mathcal{A} \models G\}$$

Problem of axiomatizability:

For which structures $\mathcal{A}$ can one axiomatize $Th(\mathcal{A})$, that is, can one write down a formula F (or a recursively enumerable set F of formulas) such that

$$Th(\mathcal{A}) = \{G \mid F \models G\}?$$

Analogously for sets of structures.

Two Interesting Theories

Let $\Sigma_{Pres} = (\{0/0, s/1, +/2\}, \emptyset)$ and $\mathbb{Z}_+ = (\mathbb{Z}, 0, s, +)$ its standard interpretation on the integers.
$Th(\mathbb{Z}_+)$ is called Presburger arithmetic (M. Presburger, 1929).
(There is no essential difference when one, instead of $\mathbb{Z}$, considers the natural numbers $\mathbb{N}$ as standard interpretation.)

Presburger arithmetic is decidable in 3EXPTIME (D. Oppen, JCSS, 16(3):323–332, 1978), and in 2EXPSPACE, using automata-theoretic methods (and there is a constant $c \geq 0$ such that $Th(\mathbb{Z}_+) \notin \text{NTIME}(2^{2^{cn}})$).

Two Interesting Theories

However, $\mathbb{N}_* = (\mathbb{N}, 0, s, +, *)$, the standard interpretation of $\Sigma_{PA} = (\{0/0, s/1, +/2, */2\}, \emptyset)$, has as theory the so-called Peano arithmetic which is undecidable, not even recursively enumerable.

Note: The choice of signature can make a big difference with regard to the computational complexity of theories.

Logical theories

Syntactic view

first-order theory: given by a set $\mathcal{F}$ of (closed) first-order Σ-formulae.

the models of $\mathcal{F}$: $\quad \mathsf{Mod}(\mathcal{F}) = \{\mathcal{A} \in \Sigma\text{-alg} \mid \mathcal{A} \models G, \text{ for all } G \text{ in } \mathcal{F}\}$

Semantic view

given a class $\mathcal{M}$ of Σ-algebras

the first-order theory of $\mathcal{M}$: $\mathsf{Th}(\mathcal{M}) = \{G \in F_\Sigma(X) \text{ closed} \mid \mathcal{M} \models G\}$

Theories

$\mathcal{F}$ set of (closed) first-order formulae

$$\mathrm{Mod}(\mathcal{F}) = \{A \in \Sigma\text{-alg} \mid \mathcal{A} \models G, \text{ for all } G \text{ in } \mathcal{F}\}$$

$\mathcal{M}$ class of Σ-algebras

$$\mathrm{Th}(\mathcal{M}) = \{G \in F_\Sigma(X) \text{ closed} \mid \mathcal{M} \models G\}$$

$\mathrm{Th}(\mathrm{Mod}(\mathcal{F}))$ the set of formulae true in all models of $\mathcal{F}$

represents exactly the set of consequences of $\mathcal{F}$

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$\mathrm{Th}(\mathrm{Mod}(\mathcal{F}))$ the set of formulae true in all models of $\mathcal{F}$
represents exactly the set of consequences of $\mathcal{F}$

Note: $\mathcal{F} \subseteq \mathrm{Th}(\mathrm{Mod}(\mathcal{F}))$ (typically strict)

$\mathcal{M} \subseteq \mathrm{Mod}(\mathrm{Th}(\mathcal{M}))$ (typically strict)

Examples

1. Groups

Let $\Sigma = (\{e/0, */2, i/1\}, \emptyset)$

Let $\mathcal{F}$ consist of all (universally quantified) group axioms:

$$\begin{array}{lll} \forall x, y, z & x * (y * z) \approx (x * y) * z & \\ \forall x & x * i(x) \approx e & \wedge \quad i(x) * x \approx e \\ \forall x & x * e \approx x & \wedge \quad e * x \approx x \end{array}$$

Every group $\mathcal{G} = (G, e_G, *_G, i_G)$ is a model of $\mathcal{F}$

$\mathsf{Mod}(\mathcal{F})$ is the class of all groups

$\mathcal{F} \subset \mathsf{Th}(\mathsf{Mod}(\mathcal{F}))$

Examples

2. Linear (positive)integer arithmetic

Let $\Sigma = (\{0/0, s/1, +/2\}, \{\leq /2\})$

Let $\mathbb{Z}_+ = (\mathbb{Z}, 0, s, +, \leq)$ the standard interpretation of integers.

$\{\mathbb{Z}_+\} \subset \mathsf{Mod}(\mathsf{Th}(\mathbb{Z}_+))$

3. Uninterpreted function symbols

Let $\Sigma = (\Omega, \Pi)$ be arbitrary

Let $\mathcal{M} = \Sigma$-alg be the class of all Σ-structures

The theory of uninterpreted function symbols is Th(Σ-alg) the family of all first-order formulae which are true in all Σ-algebras.

Examples

4. Lists

Let $\Sigma = (\{\mathsf{car}/1, \mathsf{cdr}/1, \mathsf{cons}/2\}, \emptyset)$

Let $\mathcal{F}$ be the following set of list axioms:

$$\begin{aligned} \mathsf{car}(\mathsf{cons}(x, y)) &\approx x \\ \mathsf{cdr}(\mathsf{cons}(x, y)) &\approx y \\ \mathsf{cons}(\mathsf{car}(x), \mathsf{cdr}(x)) &\approx x \end{aligned}$$

$\mathsf{Mod}(\mathcal{F})$ class of all models of $\mathcal{F}$

$\mathsf{Th}_{\mathsf{Lists}} = \mathsf{Th}(\mathsf{Mod}(\mathcal{F}))$ theory of lists (axiomatized by $\mathcal{F}$)

2.4 Algorithmic Problems

Validity(F): $\models F$?

Satisfiability(F): F satisfiable?

Entailment(F,G): does F entail G?

Model($\mathcal{A}$,F): $\mathcal{A} \models F$?

Solve($\mathcal{A}$,F): find an assignment β such that $\mathcal{A}, \beta \models F$

Solve(F): find a substitution σ such that $\models F\sigma$

Abduce(F): find G with "certain properties" such that G entails F

Decidability/Undecidability

In 1931, Gödel published his incompleteness theorems in

"Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme"

(in English "On Formally Undecidable Propositions of Principia Mathematica and Related Systems").

He proved for any computable axiomatic system that is powerful enough to describe the arithmetic of the natural numbers (e.g. the Peano axioms or Zermelo-Fraenkel set theory with the axiom of choice), that:

- If the system is consistent, it cannot be complete.
- The consistency of the axioms cannot be proven within the system.

Decidability/Undecidability

These theorems ended a half-century of attempts, beginning with the work of Frege and culminating in Principia Mathematica and Hilbert's formalism, to find a set of axioms sufficient for all mathematics.

The incompleteness theorems also imply that not all mathematical questions are computable.

Consequences of Gödel's Famous Theorems

1. For most signatures Σ, validity is undecidable for Σ-formulas.
   (One can easily encode Turing machines in most signatures.)
2. For each signature Σ, the set of valid Σ-formulas is recursively enumerable.
   (We will prove this by giving complete deduction systems.)
3. For $\Sigma = \Sigma_{PA}$ and $\mathbb{N}_* = (\mathbb{N}, 0, s, +, *)$, the theory $Th(\mathbb{N}_*)$ is not recursively enumerable.

These undecidability results motivate the study of subclasses of formulas (fragments) of first-order logic

Q: Can you think of any fragments of first-order logic for which validity is decidable?

Some Decidable Fragments/Problems

Validity/Satisfiability/Entailment: Some decidable fragments:

- Variable-free formulas without equality:
  satisfiability is NP-complete. (why?)
- Variable-free Horn clauses (clauses with at most one positive atom): entailment is decidable in linear time.
- Monadic class: no function symbols, all predicates unary;
  validity is NEXPTIME-complete.
- Q: Other decidable fragments of FOL (with variables)?
  Which methods for proving decidability?

Decidable problems.

Finite model checking is decidable in time polynomial in the size of the structure and the formula.

Goals

Identify:

- decidable fragments of first-order logic
- fragments of FOL for which satisfiability checking is easy

Methods:

- Theoretical methods (automata theory, finite model property)
- Adjust automated reasoning techniques
  (e.g. to obtaining efficient decision procedures)

  Extend methods for automated reasoning in propositional logic?

  Instantiation/reduction to propositional logic

  Extend the resolution calculus for first-order logic

Goals

Extend methods for automated reasoning in propositional logic?

Instantiation/reduction to propositional logic

Extend the resolution calculus for first-order logic

Ingredients:

- Give a method for translating formulae to clause form
- Regard formulae with variables as a set of all their instances (where variables are instantiated with ground terms)
  - Show that only certain instances are needed
    $\mapsto$ reduction to propositional logic
  - Finite encoding of infinitely many inferences
    $\mapsto$ resolution for first-order logic

2.5 Normal Forms and Skolemization

Study of normal forms motivated by

- reduction of logical concepts,
- efficient data structures for theorem proving.

The main problem in first-order logic is the treatment of quantifiers. The subsequent normal form transformations are intended to eliminate many of them.

Prenex Normal Form

Prenex formulas have the form

$$Q_1 x_1 \ldots Q_n x_n \; F,$$

where F is quantifier-free and $Q_i \in \{\forall, \exists\}$;
we call $Q_1 x_1 \ldots Q_n x_n$ the quantifier prefix and F the matrix of the formula.

Prenex Normal Form

Computing prenex normal form by the rewrite relation $\Rightarrow_P$:

$$
\begin{array}{rcll}
(F \leftrightarrow G) & \Rightarrow_P & (F \to G) \wedge (G \to F) & \\
\neg QxF & \Rightarrow_P & \overline{Q}x\neg F & (\neg Q) \\
(QxF\ \rho\ G) & \Rightarrow_P & Qy(F[y/x]\ \rho\ G),\ y \text{ fresh},\ \rho \in \{\wedge, \vee\} & \\
(QxF \to G) & \Rightarrow_P & \overline{Q}y(F[y/x] \to G),\ y \text{ fresh} & \\
(F\ \rho\ QxG) & \Rightarrow_P & Qy(F\ \rho\ G[y/x]),\ y \text{ fresh},\ \rho \in \{\wedge, \vee, \to\} &
\end{array}
$$

Here $\overline{Q}$ denotes the quantifier dual to Q, i.e., $\overline{\forall} = \exists$ and $\overline{\exists} = \forall$.

Example

$$F := (\forall x((p(x) \vee q(x,y)) \wedge \exists z\, r(x,y,z))) \rightarrow ((p(z) \wedge q(x,z)) \wedge \forall z\, r(z,x,y))$$

Example

$$F := (\forall x((p(x) \vee q(x,y)) \wedge \exists z\, r(x,y,z))) \rightarrow ((p(z) \wedge q(x,z)) \wedge \forall z\, r(z,x,y))$$

$$\Rightarrow_P \quad \exists x'\, ((p(x') \vee q(x',y)) \wedge \exists z\, r(x',y,z)) \rightarrow ((p(z) \wedge q(x,z)) \wedge \forall z\, r(z,x,y))$$

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$$\Rightarrow_P \quad \exists x' \forall z' (((p(x') \vee q(x',y)) \wedge r(x',y,z'))) \rightarrow ((p(z) \wedge q(x,z)) \wedge \forall z\, r(z,x,y$$

Example

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$$\Rightarrow_P \quad \exists x' \forall z'\, ((p(x') \vee q(x',y)) \wedge r(x',y,z')) \rightarrow \forall z''((p(z) \wedge q(x,z)) \wedge r(z'',x,$$

Example

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$$\Rightarrow_P \quad \exists x' \forall z' \forall z''(((p(x') \vee q(x',y)) \wedge r(x',y,z')) \rightarrow ((p(z) \wedge q(x,z)) \wedge r(z'',x,$$

Skolemization

Intuition: remove $\exists y$.

For this:

- we introduce a concrete choice function sk_y computing y from all the arguments y depends on (i.e. from all variables $x_1, \ldots, x_n$ which occur universally quantified before y in the quantifier prefix)
- We replace y with $\mathsf{sk}_y(x_1, \ldots, x_n)$ everywhere in the scope of $\exists y$.

Transformation $\Rightarrow_S$ (to be applied outermost, *not* in subformulas):

$$\forall x_1, \ldots, x_n \exists y F \quad \Rightarrow_S \quad \forall x_1, \ldots, x_n F[\mathsf{sk}_y(x_1, \ldots, x_n)/y]$$

where sk_y/n is a new function symbol (Skolem function).

Skolemization

Together: $F \stackrel{*}{\Rightarrow}_P \underbrace{G}_{\text{prenex}} \stackrel{*}{\Rightarrow}_S \underbrace{H}_{\text{prenex, no } \exists}$

Theorem 2.9:

Let F, G, and H as defined above and closed. Then

(i) F and G are equivalent.

(ii) $H \models G$ but the converse is not true in general.

(iii) G satisfiable (wrt. Σ-alg) $\Leftrightarrow$ H satisfiable (wrt. Σ'-Alg) where $\Sigma' = (\Omega \cup SKF, \Pi)$, if $\Sigma = (\Omega, \Pi)$.

Clausal Normal Form (Conjunctive Normal Form)

$$
\begin{array}{rcl}
(F \leftrightarrow G) & \Rightarrow_K & (F \rightarrow G) \wedge (G \rightarrow F) \\
(F \rightarrow G) & \Rightarrow_K & (\neg F \vee G) \\
\neg(F \vee G) & \Rightarrow_K & (\neg F \wedge \neg G) \\
\neg(F \wedge G) & \Rightarrow_K & (\neg F \vee \neg G) \\
\neg\neg F & \Rightarrow_K & F \\
(F \wedge G) \vee H & \Rightarrow_K & (F \vee H) \wedge (G \vee H) \\
(F \wedge \top) & \Rightarrow_K & F \\
(F \wedge \bot) & \Rightarrow_K & \bot \\
(F \vee \top) & \Rightarrow_K & \top \\
(F \vee \bot) & \Rightarrow_K & F
\end{array}
$$

These rules are to be applied modulo associativity and commutativity of $\wedge$ and $\vee$. The first five rules, plus the rule $(\neg Q)$, compute the negation normal form (NNF) of a formula.

The Complete Picture

$$
\begin{aligned}
F \quad &\Rightarrow^*_P \quad Q_1 y_1 \ldots Q_n y_n \; G && (G \text{ quantifier-free}) \\
&\Rightarrow^*_S \quad \forall x_1, \ldots, x_m \; H && (m \leq n,\ H \text{ quantifier-free}) \\
&\Rightarrow^*_K \quad \underbrace{\underbrace{\forall x_1, \ldots, x_m}_{\text{leave out}} \bigwedge_{i=1}^{k} \underbrace{\bigvee_{j=1}^{n_i} L_{ij}}_{\text{clauses } C_i}}_{F'}
\end{aligned}
$$

$N = \{C_1, \ldots, C_k\}$ is called the clausal (normal) form (CNF) of F.

Note: the variables in the clauses are implicitly universally quantified.

Theorem 2.10:

Let F be closed. Then $F' \models F$. (The converse is not true in general.)

Theorem 2.11:

Let F be closed. Then F is satisfiable iff F' is satisfiable iff N is satisfiable

Example

Given: $\exists u \forall w (\exists x (p(w, x, u) \vee \forall y (q(w, x, y) \wedge \exists z\, r(y, z))))$

Example

Given: $\exists u \forall w (\exists x (p(w, x, u) \vee \forall y \, (q(w, x, y) \wedge \exists z \, r(y, z))))$

Prenex Normal Form:

$\stackrel{*}{\Rightarrow}_P \;\; \exists u \forall w \exists x \forall y \exists z ((p(w, x, u) \vee (q(w, x, y) \wedge r(y, z))))$

Example

Given: $\exists u \forall w (\exists x (p(w, x, u) \vee \forall y\, (q(w, x, y) \wedge \exists z\, r(y, z))))$

Prenex Normal Form:

$\Rightarrow^*_P \quad \exists u \forall w \exists x \forall y \exists z ((p(w, x, u) \vee (q(w, x, y) \wedge r(y, z))))$

Skolemisation:

$\Rightarrow^*_S \quad \forall w \forall y ((p(w, \mathsf{sk}_x(w), \mathsf{sk}_u) \vee (q(w, \mathsf{sk}_x(w), y) \wedge r(y, \mathsf{sk}_z(w, y)))))$

Example

Given: $\exists u \forall w (\exists x (p(w, x, u) \vee \forall y\, (q(w, x, y) \wedge \exists z\, r(y, z))))$

Prenex Normal Form:

$\Rightarrow^*_P \;\; \exists u \forall w \exists x \forall y \exists z ((p(w, x, u) \vee (q(w, x, y) \wedge r(y, z))))$

Skolemisation:

$\Rightarrow^*_S \;\; \forall w \forall y ((p(w, \mathrm{sk}_x(w), \mathrm{sk}_u) \vee (q(w, \mathrm{sk}_x(w), y) \wedge r(y, \mathrm{sk}_z(w, y)))))$

Clause normal form:

$$\Rightarrow^*_K \;\; \forall w \forall y [(p(w, \mathrm{sk}_x(w), \mathrm{sk}_u) \vee q(w, \mathrm{sk}_x(w), y)) \wedge (p(w, \mathrm{sk}_x(w), \mathrm{sk}_u) \vee r(y, \mathrm{sk}_y(w, y)))]$$

Set of clauses:

$$\{p(w, \mathrm{sk}_x(w), \mathrm{sk}_u) \vee q(w, \mathrm{sk}_x(w), y), \quad p(w, \mathrm{sk}_x(w), \mathrm{sk}_u) \vee r(y, \mathrm{sk}_y(w, y))\}$$

Optimization

Here is lots of room for optimization since we only can preserve satisfiability anyway:

- size of the CNF exponential when done naively;
- want to preserve the original formula structure;
- want small arity of Skolem functions.

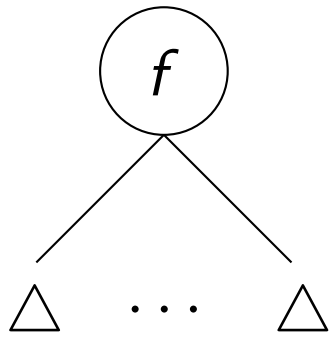
2.6 Herbrand Interpretations

From now an we shall consider PL without equality. Ω shall contains at least one constant symbol.

A Herbrand interpretation (over Σ) is a Σ-algebra $\mathcal{A}$ such that

- $U_{\mathcal{A}} = \mathsf{T}_{\Sigma}$ (= the set of ground terms over Σ)
- $f_{\mathcal{A}} : (s_1, \ldots, s_n) \mapsto f(s_1, \ldots, s_n),\ \ f/n \in \Omega$

$$f_{\mathcal{A}}(\triangle, \ldots, \triangle) =$$



Herbrand Interpretations

In other words, *values are fixed* to be ground terms and *functions are fixed* to be the term constructors. Only predicate symbols $p/m \in \Pi$ may be freely interpreted as relations $p_{\mathcal{A}} \subseteq \mathrm{T}_{\Sigma}^{m}$.

Proposition 2.12

Every set of ground atoms I uniquely determines a Herbrand interpretation $\mathcal{A}$ via

$$(s_1, \ldots, s_n) \in p_{\mathcal{A}} \quad :\Leftrightarrow \quad p(s_1, \ldots, s_n) \in I$$

Thus we shall identify Herbrand interpretations (over Σ) with sets of Σ-ground atoms.

Herbrand Interpretations

Example: $\Sigma_{Pres} = (\{0/0, s/1, +/2\},\ \{</2, \leq/2\})$

$\mathbb{N}$ as Herbrand interpretation over Σ_{Pres}:

$$
\begin{aligned}
I = \{ \quad & 0 \leq 0,\ 0 \leq s(0),\ 0 \leq s(s(0)),\ \ldots, \\
& 0 + 0 \leq 0,\ 0 + 0 \leq s(0),\ \ldots, \\
& \ldots,\ (s(0) + 0) + s(0) \leq s(0) + (s(0) + s(0)) \\
& \ldots \\
& s(0) + 0 < s(0) + 0 + 0 + s(0) \\
& \ldots\}
\end{aligned}
$$

Existence of Herbrand Models

A Herbrand interpretation I is called a Herbrand model of F, if $I \models F$.

Theorem 2.13

Let N be a set of Σ-clauses.

$$\begin{array}{lcl} N \text{ satisfiable} & \Leftrightarrow & N \text{ has a Herbrand model (over } \Sigma) \\ & \Leftrightarrow & G_\Sigma(N) \text{ has a Herbrand model (over } \Sigma) \end{array}$$

where $G_\Sigma(N) = \{C\sigma \text{ ground clause} \mid C \in N,\ \sigma : X \to \mathrm{T}_\Sigma\}$ is the set of ground instances of N.

(Proof – completeness proof of resolution for first-order logic.)

Example of a G_Σ

For Σ_{Pres} one obtains for

$$C = (x < y) \vee (y \leq s(x))$$

the following ground instances:

$(0 < 0) \vee (0 \leq s(0))$
$(s(0) < 0) \vee (0 \leq s(s(0)))$
...
$(s(0) + s(0) < s(0) + 0) \vee (s(0) + 0 \leq s(s(0) + s(0)))$
...

Consequences of Herbrans's theorem

Decidability results.

- Formulae without function symbols and without equality

  The Bernays-Schönfinkel Class $\exists^*\forall^*$

The Bernays-Schönfinkel Class

$\Sigma = (\Omega, \Pi)$, Ω is a finite set of constants

The Bernays-Schönfinkel class consists only of sentences of the form

$$\exists x_1 \ldots \exists x_n \forall y_1 \ldots \forall y_m F(x_1, \ldots, x_n, y_1, \ldots, y_n)$$

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Idea: CNF translation:

$$\begin{aligned}
&\exists \overline{x}_1 \forall \overline{y}_1 F_1 \wedge \ldots \exists \overline{x}_n \forall \overline{y}_n F_n \\
&\quad \Rightarrow_P \exists \overline{x}_1 \ldots \exists \overline{x}_n \forall \overline{y}_1 \ldots \forall \overline{y}_n F(\overline{x}_1, \ldots, \overline{x}_n, \overline{y}_1, \ldots, \overline{y}_n) \\
&\quad \Rightarrow_S \forall \overline{y}_1 \ldots \forall \overline{y}_m F(\overline{c}_1, \ldots, \overline{c}_n, \overline{y}_1, \ldots, \overline{y}_n) \\
&\quad \Rightarrow_K \forall \overline{y}_1 \ldots \forall \overline{y}_m \bigwedge \bigvee L_i((\overline{c}_1, \ldots, \overline{c}_n, \overline{y}_1, \ldots, \overline{y}_n)
\end{aligned}$$

$\overline{c}_1, \ldots, \overline{c}_n$ are tuples of Skolem constants

The Bernays-Schönfinkel Class

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Idea: CNF translation:

$$\exists \overline{x}_1 \forall \overline{y}_1 F_1 \wedge \ldots \exists \overline{x}_n \forall \overline{y}_n F_n$$
$$\Rightarrow_K^* \forall \overline{y}_1 \ldots \forall \overline{y}_m \bigwedge \bigvee L_i((\overline{c}_1, \ldots, \overline{c}_n, \overline{y}_1, \ldots, \overline{y}_n)$$

$\overline{c}_1, \ldots, \overline{c}_n$ are tuples of Skolem constants

The Herbrand Universe is finite $\mapsto$ decidability

Tractable fragments of FOL

We showed that satisfiability of any finite set of ground Horn clauses can be checked in PTIME (linear time)

Variable-free Horn clauses

Data structures

Atoms $P_1, \ldots, P_n \quad \mapsto \quad \{1, \ldots, n\}$

neg-occ-list(A): list of all clauses in which A occurs negatively

pos-occ-list(A): list of all clauses in which A occurs positively

Clause:	P_1	P_2	$\ldots$	P_n	counter
	neg	neg		pos	↑
		↑			number of literals

first-active-literal (fal): first literal not marked as deleted.

atom status:		
	pos	(deduced as positive unit clause)
	neg	(deduced as negative unit clause)
	nounit	(otherwise)

Variable-free Horn clauses

Input: Set N of Horn formulae

Step 1. Collect unit clauses; check if complementary pairs exist

forall $C \in N$ **do**

if is-unit(C) **then begin** const. time

L := first-active-literal(C) const. time

if state(atom(L)) = nounit **then** state(atom(L)) = sign(L) const. time

push(atom(L), stack)

else if state(atom(L)) $\neq$ sign(L) **then return false**

Variable-free Horn clauses

2. Process the unit clauses in the stack

while stack $\neq \emptyset$ **do**

begin A := top(stack); pop(stack)

if state(A) = pos **then** delete-literal-list := neg-oc-list(A) O(# neg-oc-list)

else delete-literal-list := pos-oc-list(A) O(# pos-oc-list)

endif

for all C in delete-literal-list **do**

if state(A) = pos **then** delete-literal(A,C) const. time + nfal - ofal

if state(A) = neg **then** delete-literal($\neg$ A,C) const. time + nfal - ofal

if unit(C) **then** L1 := first-active-literal(C) const. time

if state(atom(L1)) = nounit **then** state(atom(L1)) = sign(L1),
L1 $\rightarrow$ stack

elseif state(atom(L1)) $\neq$ sign(L1) **then** return false

endif

end

Tractable fragments of FOL

We showed that satisfiability of any finite set of ground Horn clauses can be checked in PTIME (linear time)

- Similar fragment of the Bernays-Schönfinkel class?

Motivation: Deductive Databases

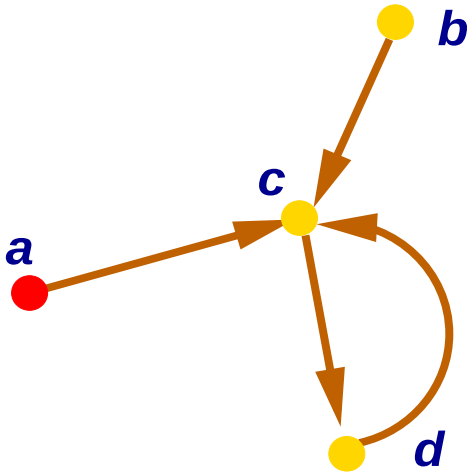
Deductive database

Inference rules:
Facts:
Query:

Motivation: Deductive Databases

Deductive database Example: reachability in graphs

Inference rules:	$\dfrac{S(x)}{R(x)}$ $\qquad$ $\dfrac{R(x) \quad E(x,y)}{R(y)}$
Facts:	$S(a), E(a,c), E(c,d), E(d,c), E(b,c)$
Query:	$R(d)$



$S(a), E(a,c), E(c,d), E(d,c), E(b,c)$

Note: S, E stored relations (Extensional DB)

R defined relation (Intensional DB)

Motivation: Deductive Databases

Deductive database Example: reachability in graphs

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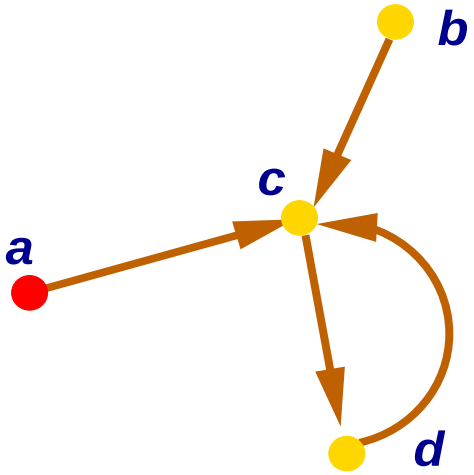

$S(a), E(a,c), E(a,d), E(c,d), E(b,c),$
$R(a)$

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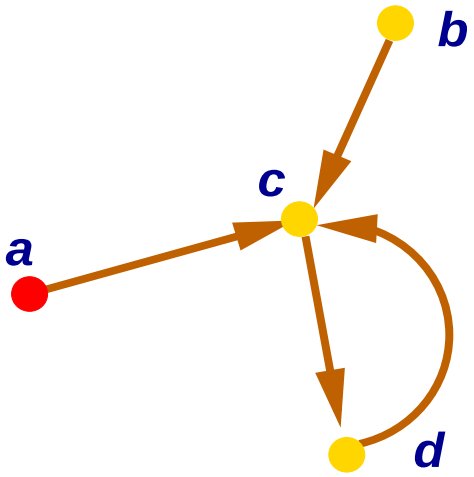
Motivation: Deductive Databases

Deductive database Example: reachability in graphs

Inference rules:	$\dfrac{S(x)}{R(x)}$ $\quad$ $\dfrac{R(x) \quad E(x,y)}{R(y)}$
Facts:	$S(a), E(a,c), E(c,d), E(d,c), E(b,c)$
Query:	$R(d)$



$S(a), E(a,c), E(a,d), E(c,d), E(b,c),$
$R(a), R(c)$

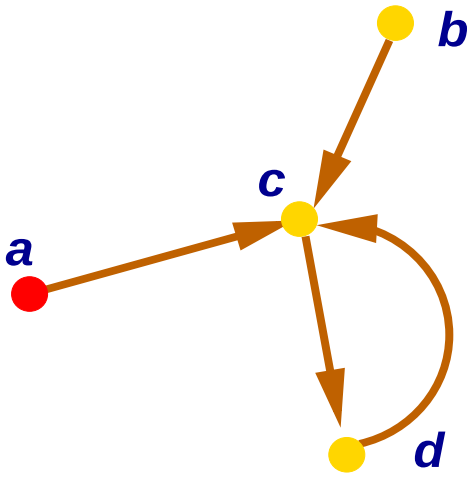
Note: S, E stored relations (Extensional DB)

R defined relation (Intensional DB)

Motivation: Deductive Databases

Deductive database Example: reachability in graphs

Inference rules:	$\dfrac{S(x)}{R(x)}$ $\qquad$ $\dfrac{R(x) \quad E(x,y)}{R(y)}$
Facts:	$S(a), E(a,c), E(c,d), E(d,c), E(b,c)$
Query:	$R(d)$



$S(a), E(a,c), E(a,d), E(c,d), E(b,c),$
$R(a), R(c), R(d)$

Note: S, E stored relations (Extensional DB)

R defined relation (Intensional DB)

Motivation: Deductive Databases

Deductive database $\mapsto$ **Datalog** (Horn clauses, no function symbols)

Inference rules:	$\underbrace{S(x) \to R(x) \qquad R(x) \wedge E(x,y) \to R(y)}_{\text{set } \mathcal{K} \text{ of Horn clauses}}$
Facts:	$\underbrace{S(a), E(a,c), E(c,d), E(d,c), E(b,c)}_{\text{set } \mathcal{F} \text{ of ground atoms}}$
Query:	$\underbrace{R(d)}_{\text{ground atom } G}$

$\mathcal{F} \models_{\mathcal{K}} G \quad$ iff $\quad \mathcal{K} \cup \mathcal{F} \models G \quad$ iff $\quad \mathcal{K} \cup \mathcal{F} \cup \neg G \models \bot$

Note: S, E stored relations (Extensional DB)

R defined relation (Intensional DB)

Motivation: Deductive Databases

Deductive database $\mapsto$ **Datalog** (Horn clauses, no function symbols)

Inference rules:	$\underbrace{S(x) \to R(x) \quad R(x) \wedge E(x,y) \to R(y)}_{\text{set } \mathcal{K} \text{ of Horn clauses}}$
Facts:	$\underbrace{S(a), E(a,c), E(c,d), E(d,c), E(b,c)}_{\text{set } \mathcal{F} \text{ of ground atoms}}$
Query:	$\underbrace{R(d)}_{\text{ground atom } G}$

Ex:

$$\dfrac{\dfrac{\dfrac{S(a) \quad S(x) \to R(x)}{R(a)} \quad E(a,c) \quad R(x) \wedge E(x,y) \to R(y)}{R(c)} \quad E(c,d) \quad R(x) \wedge E(x,y) \to R(}{R(d)}$$

Ground entailment for function-free Horn clauses

Assumption:

The signature does not contain function symbols of arity ≥ 1.

Given:

- Set H of (function-free) Horn clauses
- Ground Horn clause $G = \bigwedge A_i \rightarrow A$.

The following are equivalent:

(1) $H \models \bigwedge A_i \rightarrow A$
(2) $H \wedge \bigwedge A_i \models A$
(3) $H \wedge \bigwedge A_i \wedge \neg A \models \bot$

Decidable in PTIME in the size of G for a fixed H.

Generalization: Local theories

[McAllester,Givan'92], [Basin,Ganzinger'96,01], [Ganzinger'01]

Assumption: the signature is allowed to contain function symbols

Definition. H set of Horn clauses is called local iff for every ground clause C the following are equivalent:

(1) $H \models C$

(2) $H[C] \models C$,
where $H[C]$ is the family of all instances of H in which the variables are replaced by ground subterms occurring in H or C.

Theorem. For a fixed local theory H, testing ground entailment w.r.t. H is in PTIME.

Will be discussed in more detail in the exercises

2.7 General Resolution

Propositional resolution:

refutationally complete,

clearly inferior to the DPLL procedure
(even with various improvements).

But: in contrast to the DPLL procedure, resolution can be easily extended to non-ground clauses.

Propositional resolution: reminder

Resolution inference rule:

$$\frac{C \vee A \qquad \neg A \vee D}{C \vee D}$$

Terminology: $C \vee D$: resolvent; A: resolved atom

(Positive) factorisation inference rule:

$$\frac{C \vee A \vee A}{C \vee A}$$

Resolution for ground clauses

- Exactly the same as for propositional clauses

  Ground atoms $\mapsto$ propositional variables

Theorem

Res is sound and refutationally complete (for all sets of ground clauses)

Sample Refutation

1.	$\neg P(f(a)) \vee \neg P(f(a)) \vee Q(b)$	(given)
2.	$P(f(a)) \vee Q(b)$	(given)
3.	$\neg P(g(b,a)) \vee \neg Q(b)$	(given)
4.	$P(g(b,a))$	(given)
5.	$\neg P(f(a)) \vee Q(b) \vee Q(b)$	(Res. 2. into 1.)
6.	$\neg P(f(a)) \vee Q(b)$	(Fact. 5.)
7.	$Q(b) \vee Q(b)$	(Res. 2. into 6.)
8.	$Q(b)$	(Fact. 7.)
9.	$\neg P(g(b,a))$	(Res. 8. into 3.)
10.	$\bot$	(Res. 4. into 9.)

Resolution for ground clauses

- Refinements with orderings and selection functions:

  Need: - well-founded ordering on ground atomic formulae/literals

  - selection function (for negative literals)

  $S : C \quad \mapsto \quad$ set of occurrences of *negative* literals in C

  Example of selection with selected literals indicated as $\boxed{X}$:

  $\boxed{\neg A} \vee \neg A \vee B$

  $\boxed{\neg B_0} \vee \boxed{\neg B_1} \vee A$

Resolution Calculus $Res_S^{\succ}$

Ordered resolution with selection

$$\frac{C \vee A \qquad D \vee \neg A}{C \vee D}$$

if

1. $A \succ C$;
2. nothing is selected in C by S;
3. $\neg A$ is selected in $D \vee \neg A$,
   or else nothing is selected in $D \vee \neg A$ and $\neg A \succeq \max(D)$.

Note: For positive literals, $A \succ C$ is the same as $A \succ \max(C)$.

Ordered factoring

$$\frac{C \vee A \vee A}{(C \vee A)}$$

if A is maximal in C and nothing is selected in C.

Resolution for ground clauses

Let $\succ$ be a total and well-founded ordering on ground atoms, and S a selection function.

Theorem. $\mathrm{Res}_S^{\succ}$ is sound and refutationally complete for all sets of ground clauses.

Soundness: sufficient to show that

(1) $C \vee A, D \vee \neg A \models C \vee D$

(2) $C \vee A \vee A \models C \vee A$

Completeness: Let $\succ$ be a clause ordering, let N be saturated wrt. $Res_S^{\succ}$, and suppose that $\bot \notin N$. Then $I_N^{\succ} \models N$, where $I_N^{\succ}$ is incrementally constructed as follows:

Construction of Candidate Models Formally

Let $N, \succ$ be given.

- Order N increasing w.r.t. the extension of $\succ$ to clauses.
- Define sets I_C and Δ_C for all ground clauses C over the given signature inductively over $\succ$:

$$I_C := \bigcup_{C \succ D} \Delta_D$$

$$\Delta_C := \begin{cases} \{A\}, & \text{if } C \in N,\ C = C' \vee A,\ A \succ C',\ I_C \not\models C \\ & \text{and nothing is selected in } C \\ \emptyset, & \text{otherwise} \end{cases}$$

We say that C produces A, if $\Delta_C = \{A\}$.

The candidate model for N (wrt. $\succ$) is given as $I_N^{\succ} := \bigcup_C \Delta_C$.

(We write I_N for $I_N^{\succ}$ if $\succ$ is irrelevant or known from the context.)

Completeness

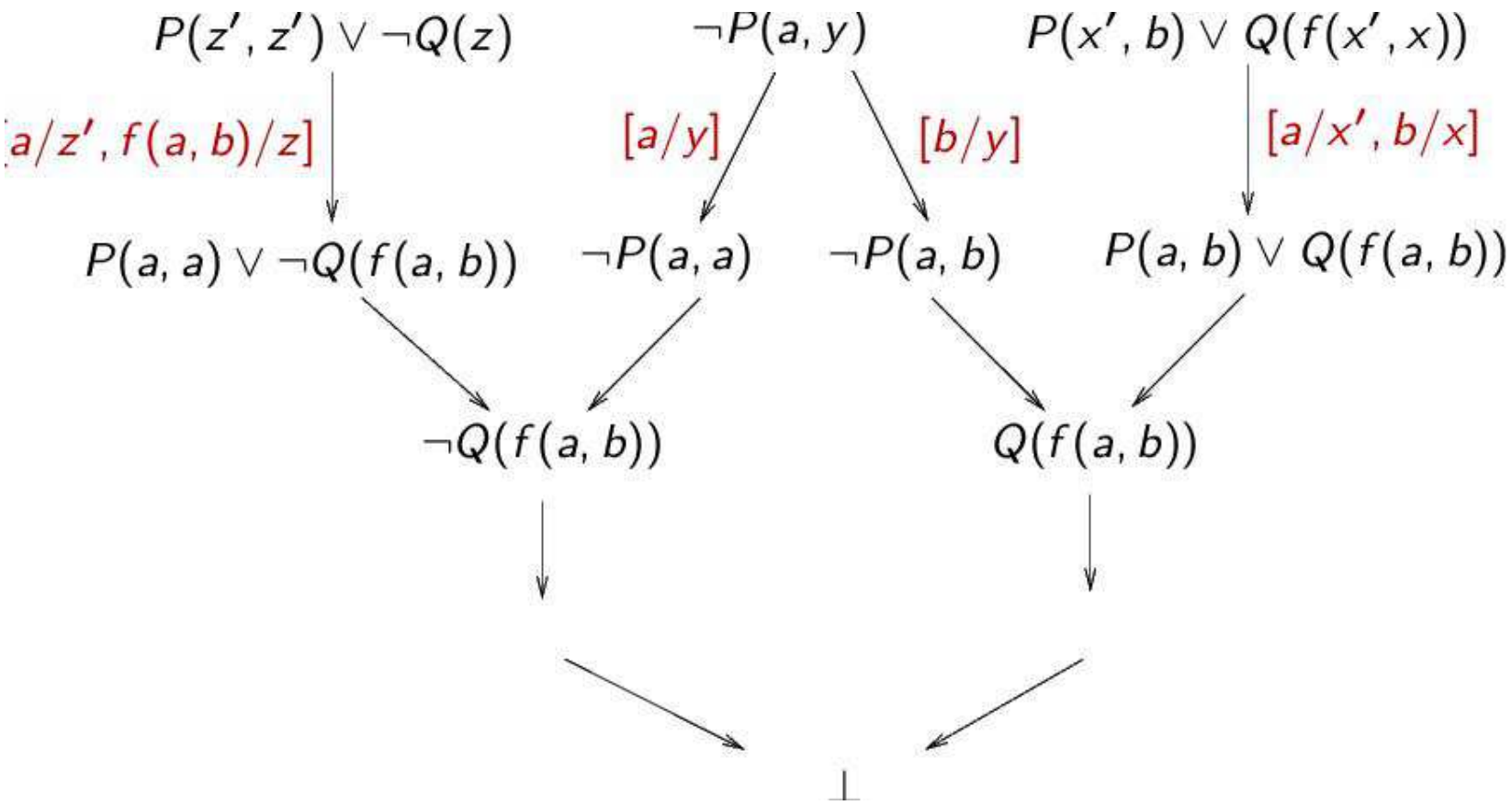
Theorem. Let $\succ$ be a clause ordering, let N be saturated wrt. $Res_S^{\succ}$, and suppose that $\bot \notin N$. Then $I_N^{\succ} \models N$.

Proof: Suppose $\bot \notin N$, but $I_N^{\succ} \not\models N$. Let $C \in N$ minimal (in $\succ$) such that $I_N^{\succ} \not\models C$. Since C is false in I_N, C is not productive. As $C \neq \bot$ there exists a maximal atom A in C.

1. $C = \neg A \vee C'$ (maximal atom occurs negatively) $\Rightarrow I_N \models A$, $I_N \not\models C'$
   Then some $D = D' \vee A \in N$ produces A. As $\frac{D' \vee A \qquad \neg A \vee C'}{D' \vee C'}$, we infer that $D' \vee C' \in N$, and $C \succ D' \vee C'$ and $I_N \not\models D' \vee C'$
   $\Rightarrow$ contradicts minimality of C.
2. $C = \boxed{\neg A} \vee C'$ ($\neg A$ is selected) $\Rightarrow I_N \models A$, $I_N \not\models C'$
   The argument in 1. applies also in this case.
3. $C = C' \vee A \vee A$. Then $\frac{C' \vee A \vee A}{C' \vee A}$ yields a smaller counterexample $C' \vee A \in N$. $\Rightarrow$ contradicts minimality of C.

General Resolution through Instantiation

Idea: instantiate clauses appropriately:



General Resolution through Instantiation

Problems:

More than one instance of a clause can participate in a proof.

Even worse: There are infinitely many possible instances.

Observation:

Instantiation must produce complementary literals
(so that inferences become possible).

Idea:

Do not instantiate more than necessary to get complementary literals.

General Resolution through Instantiation

Idea: do not instantiate more than necessary:

Lifting Principle

Problem: Make saturation of infinite sets of clauses as they arise from taking the (ground) instances of finitely many general clauses (with variables) effective and efficient.

Idea (Robinson 65):

- Resolution for general clauses:
- *Equality* of ground atoms is generalized to *unifiability* of general atoms;
- Only compute *most general* (minimal) unifiers.

Lifting Principle

Significance: The advantage of the method in (Robinson 65) compared with (Gilmore 60) is that unification enumerates only those instances of clauses that participate in an inference. Moreover, clauses are not right away instantiated into ground clauses. Rather they are instantiated only as far as required for an inference. Inferences with non-ground clauses in general represent infinite sets of ground inferences which are computed simultaneously in a single step.

Resolution for General Clauses

General binary resolution *Res*:

$$\frac{C \vee A \qquad D \vee \neg B}{(C \vee D)\sigma} \quad \text{if } \sigma = \text{mgu}(A, B) \qquad \text{[resolution]}$$

$$\frac{C \vee A \vee B}{(C \vee A)\sigma} \quad \text{if } \sigma = \text{mgu}(A, B) \qquad \text{[factorization]}$$

For inferences with more than one premise, we assume that the variables in the premises are (bijectively) renamed such that they become different to any variable in the other premises.
We do not formalize this. Which names one uses for variables is otherwise irrelevant.

Unification

Let $E = \{s_1 \doteq t_1, \ldots, s_n \doteq t_n\}$ (s_i, t_i terms or atoms) a multi-set of equality problems. A substitution σ is called a unifier of E if $s_i\sigma = t_i\sigma$ for all $1 \leq i \leq n$.

If a unifier of E exists, then E is called unifiable.

Unification after Martelli/Montanari

$$
\begin{array}{lrcl}
(1) & t \doteq t, E & \Rightarrow_{MM} & E \\
(2) & f(s_1, \ldots, s_n) \doteq f(t_1, \ldots, t_n), E & \Rightarrow_{MM} & s_1 \doteq t_1, \ldots, s_n \doteq t_n, E \\
(3) & f(\ldots) \doteq g(\ldots), E & \Rightarrow_{MM} & \bot \\
(4) & x \doteq t, E & \Rightarrow_{MM} & x \doteq t, E[t/x] \\
 & & & \quad \text{if } x \in var(E), x \notin var(t) \\
(5) & x \doteq t, E & \Rightarrow_{MM} & \bot \\
 & & & \quad \text{if } x \neq t, x \in var(t) \\
(6) & t \doteq x, E & \Rightarrow_{MM} & x \doteq t, E \\
 & & & \quad \text{if } t \notin X
\end{array}
$$

Examples

Example 1:

$$\{x \doteq f(a), g(x,x) \doteq g(x,y)\} \Rightarrow_4$$
$$\{x \doteq f(a), g(f(a),f(a)) \doteq g(f(a),y)\} \Rightarrow_2$$
$$\{x \doteq f(a), f(a) \doteq f(a), f(a) \doteq y\} \Rightarrow_1$$
$$\{x \doteq f(a), f(a) \doteq y\} \Rightarrow_6$$
$$\{x \doteq f(a), y \doteq f(a)\}$$

Example 2:

$$\{x \doteq f(a), g(x,x) \doteq h(x,y)\} \Rightarrow_3 \bot$$

Example 3:

$$\{f(x,x) \doteq f(y,g(y))\} \Rightarrow_2$$
$$\{x \doteq y, x \doteq g(y)\} \Rightarrow_4$$
$$\{x \doteq y, y \doteq g(y)\} \Rightarrow_5 \bot$$

MM: Main Properties

If $E = x_1 \doteq u_1, \ldots, x_k \doteq u_k$, with x_i pairwise distinct, $x_i \notin var(u_j)$, then E is called an (equational problem in) solved form representing the solution $\sigma_E = [u_1/x_1, \ldots, u_k/x_k]$.

Proposition 2.28:

If E is a solved form then σ_E is am mgu of E.

Theorem 2.29:

1. If $E \Rightarrow_{MM} E'$ then σ is a unifier of E iff σ is a unifier of E'
2. If $E \Rightarrow^*_{MM} \bot$ then E is not unifiable.
3. If $E \Rightarrow^*_{MM} E'$ with E' in solved form, then $\sigma_{E'}$ is an mgu of E.

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3. If $E \Rightarrow_{MM}^{*} E'$ with E' in solved form, then $\sigma_{E'}$ is an mgu of E.

Proof:
(1) We have to show this for each of the rules. Let's treat the case for the 4th rule here. Suppose σ is a unifier of $x \doteq t$, that is, $x\sigma = t\sigma$. Thus, $\sigma \circ [t/x] = \sigma[x \mapsto t\sigma] = \sigma[x \mapsto x\sigma] = \sigma$. Therefore, for any equation $u \doteq v$ in E: $u\sigma = v\sigma$, iff $u[t/x]\sigma = v[t/x]\sigma$. (2) and (3) follow by induction from (1) using Proposition 2.28.

Main Unification Theorem

Theorem 2.30:

E is unifiable if and only if there is a most general unifier σ of E, such that σ is idempotent and $dom(\sigma) \cup codom(\sigma) \subseteq var(E)$.

Proof: See e.g. Baader & Nipkow: Term rewriting and all that.

Problem: *exponential growth* of terms possible

Example:

$E = \{x_1 \approx f(x_0, x_0), x_2 \approx f(x_1, x_1), \ldots, x_n \approx f(x_{n-1}, x_{n-1})\}$

m.g.u. $[x_1 \mapsto f(x_0, x_0), x_2 \mapsto f(f(x_0, x_0), f(x_0, x_0)), \ldots]$

$x_i \mapsto$ complete binart tree of heigth i

Solution: Use acyclic term graphs; union/find algorithms

Lifting Lemma

Lemma 2.31

Let C and D be variable-disjoint clauses. If

$$\frac{\begin{array}{cc} C & D \\ \sigma\downarrow & \rho\downarrow \\ C\sigma & D\rho \end{array}}{C'} \quad \text{[propositional resolution]}$$

then there exists a substitution τ such that

$$\frac{\begin{array}{cc} C & D \end{array}}{\begin{array}{c} C'' \\ \rho\downarrow \\ C' = C''\tau \end{array}} \quad \text{[general resolution]}$$

Lifting Lemma

An analogous lifting lemma holds for factorization.

Saturation of Sets of General Clauses

Corollary 2.32:

Let N be a set of general clauses saturated under Res, i.e., $Res(N) \subseteq N$. Then also $G_\Sigma(N)$ is saturated, that is,

$$Res(G_\Sigma(N)) \subseteq G_\Sigma(N).$$

Saturation of Sets of General Clauses

Proof:

W.l.o.g. we may assume that clauses in N are pairwise variable-disjoint. (Otherwise make them disjoint, and this renaming process changes neither $Res(N)$ nor $G_\Sigma(N)$.)

Let $C' \in Res(G_\Sigma(N))$, meaning (i) there exist resolvable ground instances $C\sigma$ and $D\rho$ of N with resolvent C', or else (ii) C' is a factor of a ground instance $C\sigma$ of C.

Case (i): By the Lifting Lemma, C and D are resolvable with a resolvent C'' with $C''\tau = C'$, for a suitable substitution τ. As $C'' \in N$ by assumption, we obtain that $C' \in G_\Sigma(N)$.

Case (ii): Similar.

Herbrand's Theorem

Lemma 2.33:

Let N be a set of Σ-clauses, let $\mathcal{A}$ be an interpretation.
Then $\mathcal{A} \models N$ implies $\mathcal{A} \models G_\Sigma(N)$.

Lemma 2.34:

Let N be a set of Σ-clauses, let $\mathcal{A}$ be a *Herbrand* interpretation.
Then $\mathcal{A} \models G_\Sigma(N)$ implies $\mathcal{A} \models N$.

Herbrand's Theorem

Theorem 2.35 (Herbrand):
A set N of Σ-clauses is satisfiable if and only if it has a Herbrand model over Σ.

Proof:
The "$\Leftarrow$" part is trivial. For the "$\Rightarrow$" part let $N \not\models \bot$.

$$
\begin{aligned}
N \not\models \bot &\Rightarrow \bot \notin Res^*(N) && \text{(resolution is sound)} \\
&\Rightarrow \bot \notin G_\Sigma(Res^*(N)) \\
&\Rightarrow I_{G_\Sigma(Res^*(N))} \models G_\Sigma(Res^*(N)) && \text{(Thm. 2.23; Cor. 2.32)} \\
&\Rightarrow I_{G_\Sigma(Res^*(N))} \models Res^*(N) && \text{(Lemma 2.34)} \\
&\Rightarrow I_{G_\Sigma(Res^*(N))} \models N && (N \subseteq Res^*(N))
\end{aligned}
$$

The Theorem of Löwenheim-Skolem

Theorem 2.36 (Löwenheim–Skolem):
Let Σ be a countable signature and let S be a set of closed Σ-formulas. Then S is satisfiable iff S has a model over a countable universe.

Proof:
If both X and Σ are countable, then S can be at most countably infinite. Now generate, maintaining satisfiability, a set N of clauses from S. This extends Σ by at most countably many new Skolem functions to Σ'. As Σ' is countable, so is $T_{\Sigma'}$, the universe of Herbrand-interpretations over Σ'. Now apply Theorem 2.35.

Refutational Completeness of General Resolution

Theorem 2.37:

Let N be a set of general clauses where $Res(N) \subseteq N$. Then

$$N \models \bot \Leftrightarrow \bot \in N.$$

Proof:

Let $Res(N) \subseteq N$. By Corollary 2.32: $Res(G_\Sigma(N)) \subseteq G_\Sigma(N)$

$N \models \bot \Leftrightarrow G_\Sigma(N) \models \bot$ (Lemma 2.33/2.34; Theorem 2.35)

$\Leftrightarrow \bot \in G_\Sigma(N)$ (propositional resolution sound and complete)

$\Leftrightarrow \bot \in N$

Compactness of Predicate Logic

Theorem 2.38 (Compactness Theorem for First-Order Logic):
Let Φ be a set of first-order formulas.
Φ is unsatisfiable $\Leftrightarrow$ some finite subset $\Psi \subseteq \Phi$ is unsatisfiable.

Proof:
The "$\Leftarrow$" part is trivial. For the "$\Rightarrow$" part let Φ be unsatisfiable and let N be the set of clauses obtained by Skolemization and CNF transformation of the formulas in Φ. Clearly $Res^*(N)$ is unsatisfiable. By Theorem 2.37, $\bot \in Res^*(N)$, and therefore $\bot \in Res^n(N)$ for some $n \in \mathbb{N}$. Consequently, $\bot$ has a finite resolution proof B of depth $\leq n$. Choose Ψ as the subset of formulas in Φ such that the corresponding clauses contain the assumptions (leaves) of B.

2.12 Ordered Resolution with Selection

Motivation: Search space for *Res very* large.

Ideas for improvement:

1. In the completeness proof (Model Existence Theorem 2.23) one only needs to resolve and factor maximal atoms
   $\Rightarrow$ if the calculus is restricted to inferences involving maximal atoms, the proof remains correct
   $\Rightarrow$ *order restrictions*

2. In the proof, it does not really matter with which negative literal an inference is performed
   $\Rightarrow$ choose a negative literal don't-care-nondeterministically
   $\Rightarrow$ *selection*

Selection Functions

A selection function is a mapping

$$S : C \quad \mapsto \quad \text{set of occurrences of } \textit{negative} \text{ literals in } C$$

Example of selection with selected literals indicated as $\boxed{X}$:

$$\boxed{\neg A} \vee \neg A \vee B$$

$$\boxed{\neg B_0} \vee \boxed{\neg B_1} \vee A$$

Resolution Calculus $Res_S^{\succ}$

In the completeness proof, we talk about (strictly) maximal literals of *ground* clauses.

In the non-ground calculus, we have to consider those literals that correspond to (strictly) maximal literals of ground instances:

Let $\succ$ be a total and well-founded ordering on ground atoms.
A literal L is called [strictly] maximal in a clause C if and only if there exists a ground substitution σ such that for all L' in C: $L\sigma \succeq L'\sigma$ [$L\sigma \succ L'\sigma$].

Resolution Calculus $Res_S^{\succ}$

Let $\succ$ be an atom ordering and S a selection function.

$$\frac{C \vee A \qquad \neg B \vee D}{(C \vee D)\sigma}$$ [ordered resolution with selection]

if $\sigma = \mathrm{mgu}(A, B)$ and

(i) $A\sigma$ strictly maximal wrt. $C\sigma$;

(ii) nothing is selected in C by S;

(iii) either $\neg B$ is selected,
or else nothing is selected in $\neg B \vee D$ and $\neg B\sigma$ is maximal in $D\sigma$.

Resolution Calculus $Res_S^{\succ}$

$$\frac{C \vee A \vee B}{(C \vee A)\sigma} \qquad \text{[ordered factoring]}$$

if $\sigma = \mathrm{mgu}(A, B)$ and $A\sigma$ is maximal in $C\sigma$ and nothing is selected in C.

Soundness and Refutational Completeness

Theorem 2.39:

Let $\succ$ be an atom ordering and S a selection function such that $Res_S^{\succ}(N) \subseteq N$. Then

$$N \models \bot \Leftrightarrow \bot \in N$$

Proof:

The "$\Leftarrow$" part is trivial. For the "$\Rightarrow$" part consider first the propositional level: Construct a candidate model I_N as for unrestricted resolution, except that clauses C in N that have selected literals are not productive, even when they are false in I_C and when their maximal atom occurs only once and positively.

The result for general clauses follows using the lifting lemma.