Bradley & Manna "The Calculus of Computation"

$\boxed{\text{Examples 10.5 / 10.7}}$ (pages 272/273)

$T_1 = LI(\mathbb{Z})$ $\quad T_2 = UIF\{f\}$.

$$\varphi := 1 \leq x \wedge x \leq 2 \wedge f(x) \not\simeq f(\underbrace{1}_{w_1}) \wedge f(x) \not\simeq f(\underbrace{2}_{w_2}).$$

1. Purification:

$LI(\mathbb{Z})$	$UIF\{f\}$
$1 \leq x$	$f(x) \not\simeq f(w_1)$
$x \leq 2$	$f(x) \not\simeq f(w_2)$.
$w_1 = 1$	
$w_2 = 2$	
φ_1	φ_2

- Shared variables: $V = \{x, w_1, w_2\}$

$\boxed{\varphi \text{ is equisatisfiable to } \varphi_1 \wedge \varphi_2}$

2. Guessing

$$\alpha(V, E) = \bigwedge_{\substack{u,v \in V \\ (u,v) \in E}} u \simeq v \wedge \bigwedge_{\substack{u,v \in V \\ (u,v) \notin E}} u \not\simeq v$$ when E is an equiv. relation on V.

φ is $T_1 \cup T_2$-satisfiable iff there exists an equivalence relation E on V such that
- $\varphi_1 \wedge \alpha(V, E)$ is T_1-satisfiable and
- $\varphi_2 \wedge \alpha(V, E)$ is T_2-satisfiable.

We consider all possible equivalence relations on $\{x, w_1, w_2\}$
→ we list them by stating the partitions.

1. $E = \{\{x, w_1, w_2\}\} \mapsto \alpha(V, E) = x \simeq w_1 \wedge x \simeq w_2 \wedge w_1 \simeq w_2$.
2. $E = \{\{x, w_1\}, \{w_2\}\} \mapsto \alpha(V, E) = x \simeq w_1 \wedge x \not\simeq w_2 \wedge w_1 \not\simeq w_2$
3. $E = \{\{x, w_2\}, \{w_1\}\} \mapsto \alpha(V, E) = x \simeq w_2 \wedge x \not\simeq w_1 \wedge w_2 \not\simeq w_1$.

   In these cases $\varphi_2 \wedge \alpha(V, E)$ is unsatisfiable: $f(x) \not\simeq f(w_i)$ $i = 1 \& 2$
   it cannot be the case that $x \simeq w_i$ but

4. $E = \{\{x\}, \{w_1, w_2\}\} \mapsto \alpha(V, E) = w_1 \simeq w_2 \wedge x \not\simeq w_1 \wedge x \not\simeq w_2$.

   Then $\varphi_1 \wedge \alpha(V, E)$ is unsatisfiable: it cannot be the case that $w_1 \simeq w_2 \wedge w_1 \simeq 1 \wedge w_2 \simeq 1$ 2

5. $E = \{\{x\}, \{w_1\}, \{w_2\}\} \mapsto \alpha(V, E) = x \not\simeq w_1 \wedge x \not\simeq w_2 \wedge w_1 \not\simeq w_2$.

   Then $\varphi_1 \wedge \alpha(V, E)$ is unsatisfiable: we cannot have $x \not\simeq \overset{1}{w_1}, x \not\simeq \overset{2}{w_2}$ and $1 \leq x \leq 2$.

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Examples 10.6 / 10.9 (pages 272/274) $T_1 = LI(\mathbb{Z})$, $T_2 = UIF$

$$\boxed{\varphi := \underbrace{f(x)}_{w_1} \simeq x+y \wedge x \leq y+z \wedge x+z \leq y \wedge y=1 \wedge f(x) \neq f(\underbrace{z}_{w_2})}.$$

1. Purification

$LI(\mathbb{Z})$	$UIF_{\{f\}}$
$w_1 \simeq x+y$	$w_1 \simeq f(x)$
$x \leq y+z$	$f(x) \neq f(w_2)$.
$x+z \leq y$.	
$y=1$	
$w_2 \simeq z$	
φ_1	φ_2

2. Guessing

$V := shared(\varphi_1, \varphi_2) = \{x, w_1, w_2\}$.

we attempt to construct an arrangement,
we do not list all arrangements.

1. Suppose $x \simeq w_1$. Then $w_1 \simeq x+y$ implies $y=0$, but φ_1 contains $y=1$. Contradiction.

   $\boxed{\text{Hence: } x \neq w_1.}$

2. $\varphi_1 \wedge x \neq w_1$ and $\varphi_2 \wedge x \neq w_1$ are T_1 resp. T_2-satisfiable.

3. Suppose $x \simeq w_2$. Then $f(x) \simeq f(w_2)$, but φ_2 contains $f(x) \neq f(w_2)$ contradiction

   $\boxed{\text{Hence } x \neq w_2.}$

4. $\varphi_1 \wedge x \neq w_1 \wedge x \neq w_2$ and $\varphi_2 \wedge x \neq w_1 \wedge x \neq w_2$ are T_1/T_2-satisf.

5. Suppose $w_1 \simeq w_2$. No contradiction found.

We discovered the arrangement $\boxed{\underbrace{x \neq w_1 \wedge x \neq w_2 \wedge w_1 \simeq w_2}_{arr(V, \in)}}$
such that

$\varphi_1 \wedge arr(V, \in)$ sat. wrt T_1
$\varphi_2 \wedge arr(V, \in)$ sat. wrt T_2

Therefore φ is $T_1 \cup T_2$-satisfiable.