## Universität Koblenz-Landau

**FB 4 Informatik**


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19.1.2023

# Collection of exercises: Part 2

**Exercise 1.** Check the satisfiability of the following formulae using the DAG version of the Congruence Closure algorithm presented in the class:

1. $f(a, b) \approx f(b, a) \wedge f(c, a) \not\approx f(b, c)$
2. $f(g(a)) \approx g(f(a)) \wedge f(g(f(b))) \approx a \wedge f(b) \approx a \wedge g(f(a)) \not\approx a$
3. $f(f(f(a))) \approx f(a) \wedge f(f(a)) \approx a \wedge f(a) \not\approx a$

**Exercise 2.**

(1a) Check the satisfiability over $\mathbb{Z}$ of the following set of constraints in positive difference logic using the method presented in the lecture. In case of satisfiability find a satisfiable assignment.

(a) $x - y \leq 4 \wedge y - z \leq 2 \wedge x - z \leq 2 \wedge z - x \leq -3$

(b) $x - y \leq 4 \wedge y - z \leq 0 \wedge x - z \leq 4 \wedge z - x \leq -3 \wedge x - u \leq -4$

(1b) Check the satisfiability over $\mathbb{Z}$ of the following set of constraints in difference logic using the method presented in the lecture. In case of satisfiability find a satisfiable assignment.

(a) $x - y \leq 4 \wedge y - z \leq 0 \wedge x - z < 4 \wedge z - x \leq -3 \wedge x - u \leq -4$

(a) $x - y \leq 4 \wedge y - z \leq 0 \wedge x - z < 4 \wedge z - x < -3 \wedge x - u \leq -4$

(2a) Check the satisfiability over $\mathbb{Q}$ of the following sets of constraints in positive difference logic. In case of satisfiability find a satisfiable assignment.

(a) $x - y \leq 5 \wedge y - u \leq 4 \wedge x - z \leq -1 \wedge z - x \leq 1.$

(b) $x - y \leq 5 \wedge y - u \leq 4 \wedge x - z \leq -1 \wedge z - x \leq 1 \wedge z - y \leq -5.$

(2a) Check the satisfiability over $\mathbb{Q}$ of the following sets of constraints in difference logic. In case of satisfiability find a satisfiable assignment.

(a) $x - y \leq 5 \wedge y - u \leq 4 \wedge x - z < -1 \wedge z - x \leq 1.$

(b) $x - y \leq 5 \wedge y - u \leq 4 \wedge x - z < -0.5 \wedge z - x < 1 \wedge z - y \leq -5.$

**Exercise 3.** Check the satisfiability of the following $LI(\mathbb{Q})$-formulae using the Fourier-Motzkin method for quantifier elimination:

- $F_1: \quad -x_1 + x_2 \leq 3 \quad \wedge \quad x_1 - 5 \leq x_3 \quad \wedge \quad x_3 - x_1 - x_2 \leq 4$
- $F_2: \quad -x_1 + x_2 \leq 3 \quad \wedge \quad x_1 - 5 \leq x_3 \quad \wedge \quad x_3 - x_1 - x_2 = 4$

**Exercise 4.** Check the satisfiability of the following $LI(\mathbb{Q})$-formula using the Weisspfenning-Loos method for quantifier elimination:

$$F: \quad x_1 + 2x_2 \leq 3 \quad \wedge \quad x_2 - x_1 \leq 4.$$

**Exercise 5.** Use the Nelson-Oppen procedure for checking the satisfiability of the following formulae:

1. $1 \leq c \wedge c \leq 3 \wedge f(c) \not\approx f(1) \wedge f(c) \not\approx f(3) \wedge f(1) \not\approx f(2)$
   in the combination $LI(\mathbb{Z}) \cup UIF_{\{f\}}$.
2. $f(c) \approx c + d \wedge c \leq d + e \wedge c + e \leq d \wedge d = 1 \wedge f(c) \not\approx f(2)$
   in the combination $LI(\mathbb{Z}) \cup UIF_{\{f\}}$.
3. $c + d \approx e \wedge f(e) \approx e \wedge f(c + d) \not\approx e$
   in the combination $LI(\mathbb{Q}) \cup UIF_{\{f\}}$.

**Exercise 6:** Let $\mathcal{T}$ be the combination of $LI(\mathbb{Q})$ (linear arithmetic over $\mathbb{Q}$) and $UIF_\Sigma$, the theory of uninterpreted function symbols in a signature $\Sigma$ containing the unary function $h$.

Check the satisfiability of the following ground formula w.r.t. $\mathcal{T}$ using the deterministic version of the Nelson-Oppen procedure (after purifying the formulae check for entailment of equalities between shared constants and propagate the entailed equalities):

- $\phi: \quad a + h(h(h(b)) + c) \approx e \quad \wedge \quad h(b) \approx b' \quad \wedge \quad e \approx h(b') + c \quad \wedge \quad a + h(e) \not\approx e.$

For reasoning in $UIF_\Sigma$ use the graph-based method for computing the congruence closure presented in the class.

**Exercise 7.** Let $\mathcal{T} = LI(\mathbb{Q})$, and let $Q := x \leq 10, R := y \leq x, P := x + x + x \leq 30, S := y \geq 10$. Use a DPLL($\mathcal{T}$) method to check the satisfiability w.r.t. $\mathcal{T}$ of the following set of clauses:

$$\begin{array}{ll} (1) & \neg R \vee P \\ (2) & \neg Q \vee \neg P \\ (3) & R \vee P \\ (4) & S \end{array}$$

For checking the satisfiability of conjunctions of inequalities in $LI(\mathbb{Q})$ use the Fourier-Motzkin method.

**Exercise 8.** In what follows we consider the theory of arrays defined in the lecture. We assume that the theory of indices $\mathcal{T}_i$ is $LI(\mathbb{Z})$, and the theory of elements $\mathcal{T}_e$ is $LI(\mathbb{Q})$.

Which of the formulae below are in the array property fragment and which are not? Justify your answer. (The universally quantified variables $i, j$ are of sort index; the indices $k, l_i, u_i, i = 1, 2$ which are not universally quantified are considered to be constants of sort index)

(1) $\forall i\ (a[a[i]] > a[i])$

(2) $\forall i\ (i > a[i])$

(3) $\forall i\ (a[i] > b[i])$

(4) $\forall i\ (i \leq a[k] \rightarrow a[i] = a[k])$

(5) $\forall i, j\ (l_1 < i < u_1 < l_2 < j < u_2 \rightarrow a[i] \leq a[j])$

(6) $\forall i, j\ (l_1 < i < j < u_2 \rightarrow a[i] \leq a[j])$

(7) $\forall i, j\ (l_1 < i \leq j < u_2 \rightarrow a[i] \leq a[j])$

**Exercise 9.**

(**Note:** the probability that such an exercise would come up in the exam is extremely low)

Consider the array property formula:

$$F : \mathit{write}(a, l, v_1)[k] = b[k] \wedge b[k] = v_2 \wedge a[k] = v_1 \wedge v_1 \neq v_2 \wedge \forall i (i \leq l-1 \rightarrow a[i] = b[i]) \wedge \forall i (i \geq l+1 \rightarrow a[i] = b[i])$$

(1) Apply Steps 1–6 described in the lecture to $F$. Let $F_6$ be the formula obtained after Step 6.

(2) Check the satisfiability of $F_6$ using one of the versions of the $DPLL(\mathcal{T})$ procedure presented in the class. For theory reasoning in combinations of theories use the Nelson-Oppen procedure.