Universität Koblenz-Landau

FB 4 Informatik

Prof. Dr. Viorica Sofronie-Stokkermans

October 31, 2022

Exercises for "Decision Procedures for Verification"
Exercise sheet 2

Note: Exercise 2.1 and the supplementary exercise were already on Exercise sheet 1 for "Non-Classical Logics" and were discussed in the first exercise session for that lecture or will be discussed on 7.11. Exercise 2.2 is similar to an exercise on Exercise sheet 1 for "Non-Classical Logics". If you attended that exercise session you do not need to solve them also for "Decision Procedures for Verification".

Exercise 2.1: *(2 P)*
Determine the polarity of the following subformulae of

$$F = \neg((\neg(P \wedge \neg Q)) \vee (R \vee \neg S)) \vee (U \wedge V)$$

(1) $(P \wedge \neg Q)$

(2) Q

(3) $(R \vee \neg S)$

(4) S

(5) V

(6) $((\neg(P \wedge \neg Q)) \vee (R \vee \neg S))$

Exercise 2.2: *(4 P)*
Let F be the following formula:

$$\neg[((Q \wedge \neg P) \vee \neg(Q \vee R)) \rightarrow ((Q \rightarrow P) \wedge (Q \wedge \neg P))] \wedge (P \vee R)$$

(1) Compute the negation normal form (NNF) F' of F.

(2) Convert F' to CNF using the satisfiability-preserving transformation described in the lecture.

Exercise 2.3: *(3 P)*
Let $\succ$ be the ordering on propositional variables defined by $P \succ Q \succ R \succ S \succ U$. Use the ordered resolution calculus $Res^{\succ}$ to prove that the following set of clauses is unsatisfiable:

(1)	$\neg Q \vee R \vee \neg P \vee \neg U$
(2)	$\neg Q \vee \neg P \vee S$
(3)	$P \vee \neg Q$
(4)	$\neg S \vee \neg R$
(5)	Q
(6)	$R \vee U$

Exercise 2.4: (*2 P*)
Assume $S \succ P \succ Q \succ R$. Let N be the following set of clauses:

(1)	$\neg Q \vee \neg P$
(2)	$R \vee P$
(3)	$Q \vee S$
(4)	$\neg Q \vee \neg S$

(1) Which literals are maximal in the clauses of N?

(2) Which inferences are possible in the ordered resolution calculus $\mathsf{Res}^{\succ}$ with the rules:

$$\frac{C \vee A \qquad D \vee \neg A}{C \vee D} \qquad \text{[ordered resolution]}$$

if C, D are clauses and A is a propositional variable with:
(i) $A \succ C$ (A is larger (in $\succ$) than the maximal literal in C);
(ii) $\neg A \succeq \max(D)$ (i.e. $\neg A$ is larger than or equal to the maximal literal of D).

$$\frac{C \vee A \vee A}{(C \vee A)} \qquad \text{[ordered factoring]}$$

if C is a clause and A a propositional variable such that A is maximal in C.

- Let S be the selection function which selects the negative literal $\neg Q$ in the clauses (1) and (4). Which inferences are possible in the ordered resolution calculus with selection $\mathsf{Res}_S^{\succ}$ presented in the lecture.

Exercise 2.5: (*1 P*)
Find a total ordering on the propositional variables A, B, C, D, E, such that the associated clause ordering $\succ_C$ orders the clauses like this:

$$B \vee C \succ_C A \vee A \vee \neg C \succ_C C \vee E \succ_C C \vee D \succ_C \neg A \vee D \succ_C \neg E.$$

Exercise 2.6: *(4 P)*
Let N be the following set of clauses:

$$
\begin{array}{ll}
(1) & \neg P_3 \vee P_1 \vee P_1 \\
(2) & \neg P_2 \vee P_1 \\
(3) & P_4 \vee P_4 \\
(4) & P_4 \\
(5) & P_3 \vee \neg P_2 \\
(6) & P_4 \vee P_3
\end{array}
$$

(1) Let $\succ$ be the ordering on propositional variables defined by $P_4 \succ P_3 \succ P_2 \succ P_1$. Sort the clauses in N according to $\succ_C$. Which literals are maximal in the clauses of N?
(2) Define a selection function S such that N is saturated under $Res_S^{\succ}$.
(3) Construct a model of N using the canonical construction presented in the lecture.

Supplementary exercises (to be discussed in one of the following exercise sessions)

Exercise 2.7: *(2 P)*
Let F be a formula, P a propositional variable not occurring in F, and F' a subformula of F. Prove: The formula $F[P] \wedge (P \leftrightarrow F')$ is satisfiable if and only if $F[F']$ is satisfiable.

Here $F[F']$ is the formula F (in which F' was not replaced) and $F[P]$ is obtained from the formula F by replacing the subformula F' with the propositional variable P.

Hint: You can first prove (by induction over the formula structure of F) that for any valuation $\mathcal{A}$, if $\mathcal{A}(P) = \mathcal{A}(F')$ then $\mathcal{A}(F[P]) = \mathcal{A}(F[F'])$. This result is then used to prove the claim.

Exercise 2.8: *(4 P)*
Let F be a formula containing neither $\rightarrow$ nor $\leftrightarrow$, P a propositional variable not occurring in F, and F' a subformula of F. Prove:

- If F' has positive polarity in F then $F[F']$ is satisfiable if and only if $F[P] \wedge (P \rightarrow F')$ is satisfiable.
- If F' has negative polarity in F then $F[F']$ is satisfiable if and only if $F[P] \wedge (F' \rightarrow P)$ is satisfiable.

Please submit your solution until Monday, November 7, 2022 at 23:59. Joint solutions prepared by up to three persons are allowed. Please do not forget to write your name on your solution.

Submission possibilities:

- Homework-02 directory in OLAT

Reminder: The structural induction principle (for propositional logic).

Variant 1

Let $\mathcal{B}$ be a property of formulae in propositional logic. We want to prove that every formula over a set Π of propositional variables has property $\mathcal{B}$.

For this we proceed as follows:

Induction basis. We prove that:

- for every propositional variable $P \in \Pi$, P has property $\mathcal{B}$;
- $\bot$ and $\top$ have property $\mathcal{B}$.

Let F be a formula with propositional variables in Π such that $F \not\in \Pi, F \neq \bot, F \neq \top$.

Induction hypothesis. We assume that every strict subformula G of F has property $\mathcal{B}$.

Induction step. We use the induction hypothesis to show that also F has property $\mathcal{B}$.

We distinguish the following cases:

- $F = \neg F_1$
- $F = F_1 \ \mathsf{op}\ F_2$ for $\mathsf{op} \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$

Then we can conclude that property $\mathcal{B}$ holds for all Π-formulae.

Variant 2

Let $\mathcal{B}$ be a property of formulae in propositional logic. Assume that the following hold:

- for every propositional variable $P \in \Pi$, P has property $\mathcal{B}$;
- $\bot$ and $\top$ have property $\mathcal{B}$;
- if $F = F_1 \ \mathsf{op}\ F_2$ for $\mathsf{op} \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$ and if both F_1 and F_2 have property $\mathcal{B}$ then F has property $\mathcal{B}$;
- if $F = \neg F_1$ and F_1 has property $\mathcal{B}$ then F has property $\mathcal{B}$.

Then property $\mathcal{B}$ holds for all Π-formulae.