Universität Koblenz-Landau

FB 4 Informatik

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Exercises for "Decision Procedures for Verification"
Exercise sheet 8

Exercise 8.1: *(2 P)*
To which of the classes discussed in the lecture (the Bernays-Schönfinkel class, the Ackermann class or the monadic class) do the following formulae belong:

(1) $\exists y \forall x \ \ ((p(x) \vee r(x, y)) \wedge q(y))$
(2) $\forall x \exists y \forall z \exists u((p(x) \vee q(y)) \wedge (q(y) \vee p(u))$
(3) $\exists z \forall x \exists y(p(x) \vee q(y)) \wedge q(z)$
(4) $\exists x \forall y(P(x) \vee R(y)) \wedge Q(y)$

Note that they can be in more than one, or in none of the classes above.

Exercise 8.2: *(4 P)*
Let ϕ be the following (ground) formula over the signature $\Sigma = (\{f/1, c/0, d/0\}, \{\approx\})$:

$$f(f(c)) \approx f(c) \wedge f(f(c)) \approx f(d) \wedge d \not\approx f(c).$$

(1) Compute $FLAT(\phi)$ (the formula obtained by recursively replacing, in a bottom-up fashion, any term of the form $f(c')$, where c' is a constant, with a new constant).
(2) Compute $FC(\phi)$ (the set of functional consistency axioms associated with the flattening above):
$$FC(\phi) = \{c_1 \approx c_2 \rightarrow d_1 \approx d_2 \mid d_i \text{ is introduced as an abbreviation for } f(c_i)\}.$$
(3) Check whether $FLAT(\phi) \wedge FC(\phi)$ is satisfiable.
(4) Is ϕ is satisfiable? Justify your answer.

Exercise 8.3: *(6 P)*
Check the satisfiability of the following ground formulae using the algorithm based on congruence closure presented in the lecture.

(1) $\phi_1 = f(f(c)) \approx f(c) \wedge f(f(c)) \approx f(d) \wedge d \not\approx f(c)$.
(2) $\phi_2 = f(f(c)) \approx f(c) \wedge f(c) \approx d \wedge f(d) \not\approx f(f(c))$.

Supplementary exercises:

Exercise 8.4: *(3 P)*
Prove the $\Rightarrow$ part in the correctness proof of the algorithm for checking the validity of a conjunction of literals in UIF, under the assumption that an algorithm for computing the congruence closure of a set R of pairs of vertices in a graph G exists.

Let $\phi := \bigwedge_{i=1}^{n} s_i \approx t_i \wedge \bigwedge_{j=1}^{m} s'_j \not\approx t'_j$ be a ground formula. Let $G = (V, E)$ be the labelled directed graph constructed from ϕ as in the description of the congruence closure algorithm (Solution 3). Let $R = \{(v_{s_i}, v_{t_i}) \mid i \in \{1, \ldots, n\}\}$, and let R^c be the congruence closure of R.

(1) $\mathcal{A}$ is a Σ-structure such that $\mathcal{A} \models \phi$. Prove that $[v_s]_{R^c} = [v_t]_{R^c}$ implies that $\mathcal{A} \models s \approx t$.
(2) Assume that ϕ is satisfiable. Prove that $[v_{s'_j}]_{R^c} \neq [v_{t'_j}]_{R^c}$.

Hint: Use the fact that if $[v_s]_{R^c} = [v_t]_{R^c}$ then there is a derivation for $(v_s, v_t) \in R^c$ in the calculus defined before; use induction on the length of derivation to show that $\mathcal{A} \models s \approx t$.

Exercise 8.5: *(3 P)*
Let F be a closed first-order formula with equality over a signature $\Sigma = (\Omega, \Pi)$. Let $\sim \notin \Omega$ be a new binary relation symbol (written as an infix operator). Let the set $Eq(\Sigma)$ contain the formulas

$$\forall x\,(x \sim x)$$
$$\forall x, y\,(x \sim y \rightarrow y \sim x)$$
$$\forall x, y, z\,(x \sim y \wedge y \sim z \rightarrow x \sim z)$$

and for every $f/n \in \Omega$ the formula

$$\forall x_1, \ldots, x_n, y_1, \ldots, y_n\,(x_1 \sim y_1 \wedge \cdots \wedge x_n \sim y_n \rightarrow f(x_1, \ldots, x_n) \sim f(y_1, \ldots, y_n))$$

and for every $p/n \in \Pi$ the formula

$$\forall x_1, \ldots, x_n, y_1, \ldots, y_n\,(x_1 \sim y_1 \wedge \cdots \wedge x_n \sim y_n \wedge p(x_1, \ldots, x_n) \rightarrow p(y_1, \ldots, y_n)).$$

Let $\tilde{F}$ be the formula that one obtains from F if every occurrence of the equality symbol $\approx$ is replaced by the relation symbol $\sim$.

(a) *Definition.* A binary relation $\sim$ on the support of a Σ-algebra satisfying all properties in $Eq(\Sigma)$ is called a congruence relation.

Let $\mathcal{A}$ be a model of $\tilde{F} \cup Eq(\Sigma)$. Show that the interpretation $\sim_{\mathcal{A}}$ of $\sim$ in $\mathcal{A}$ is a congruence relation. (It is enough if you prove *one* of the properties of congruence relations, say symmetry; the other properties are proved analogously.)

(b) Let $\mathcal{A}$ be a model of $\tilde{F} \cup Eq(\Sigma)$. Use the congruence relation $\sim_{\mathcal{A}}$ to construct a model of F and prove that it is a model.

(c) Prove that a formula F is satisfiable if and only if $Eq(\Sigma) \cup \{\tilde{F}\}$ is satisfiable.

Please submit your solution until Wednesday, January 3, 2023 at 12:00. Joint solutions prepared by up to three persons are allowed. Please do not forget to write your name on your solution.

Submission possibilities:

- Use the directory Homework 08 in OLAT