Decision Procedures in Verification

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Motivation

Long-term goal of research in computer science

- use computers as 'intelligent assistants' in

  e.g. mathematics, engineering (and other fields)

Main problem

- complex description of problems to be solved

  $\mapsto$ complex systems, complex encoding

Examples of application domains

MATHEMATICS

Tasks

- construct proofs
- check proofs

Theories

- numbers
- polynomials
- functions over numeric domains
- algebras

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Example: Lipschitz functions

$$\mathbb{R} \cup (\mathsf{L}^{f}_{c,\lambda_1}) \cup (\mathsf{L}^{g}_{c,\lambda_2}) \models (\mathsf{L}^{f+g}_{c,(\lambda_1+\lambda_2)})$$

$(\mathsf{L}^{f}_{c,\lambda_1})$ $\forall x\ |f(x) - f(c)| \leq \lambda_1 \cdot |x - c|$

$(\mathsf{L}^{g}_{c,\lambda_2})$ $\forall x\ |g(x) - g(c)| \leq \lambda_2 \cdot |x - c|$

$(\mathsf{L}^{f+g}_{c,(\lambda_1+\lambda_2)})$ $\forall x\ |f(x)+g(x)-f(c)-g(c)| \leq (\lambda_1+\lambda_2) \cdot |x-c|$

Similar:
- free functions; (piecewise) monotone functions
- functions defined according to a partition of their domain of definition, ...

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VERIFICATION

Tasks
- **reactive and hybrid systems**
  - safety / lifeness
- **programs**
  - correctness
  - termination

Infinite state systems (software, real time, hybrid)

- simulation/testing cannot guarantee absence of errors
  $\mapsto$ need symbolic methods

Solution: - Build 'formal model' of the system;
- Prove that properties are 'consequences of the model'

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Theories
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- data types
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Example: Does BUBBLESORT return a sorted array?

```
int [] BubbleSort(int[] a) {
  int i, j, t;
  for (i := |a| − 1; i > 0; i := i − 1) {
    for (j := 0; j < i; j := j + 1) {
      if (a[j] > a[j + 1]){t := a[j];
                           a[j] := a[j + 1];
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}} return a}
```

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correctness/termination
- reactive/hybrid systems

safety/lifeness
- cryptography

correctness crypt. prot.

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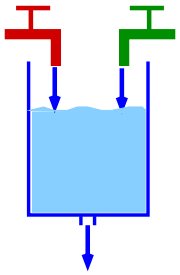
Controllers





Check:
- No overflow
- Substances in the right proportion
- If substances in wrong proportion, tank can be drained in $\leq$ 200s.

Determine values for parameters such that this is the case

Examples of application domains

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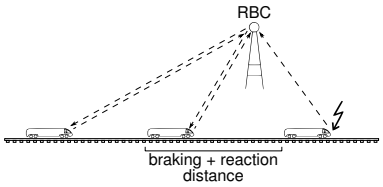
Controllers



Train controllers



- **Task:** check collision freeness

Examples of application domains

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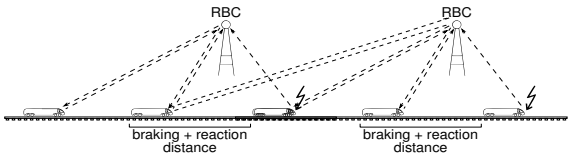
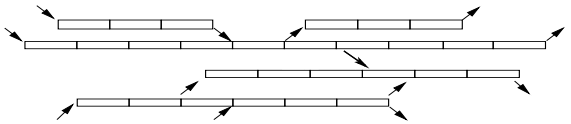
Two or more controllers

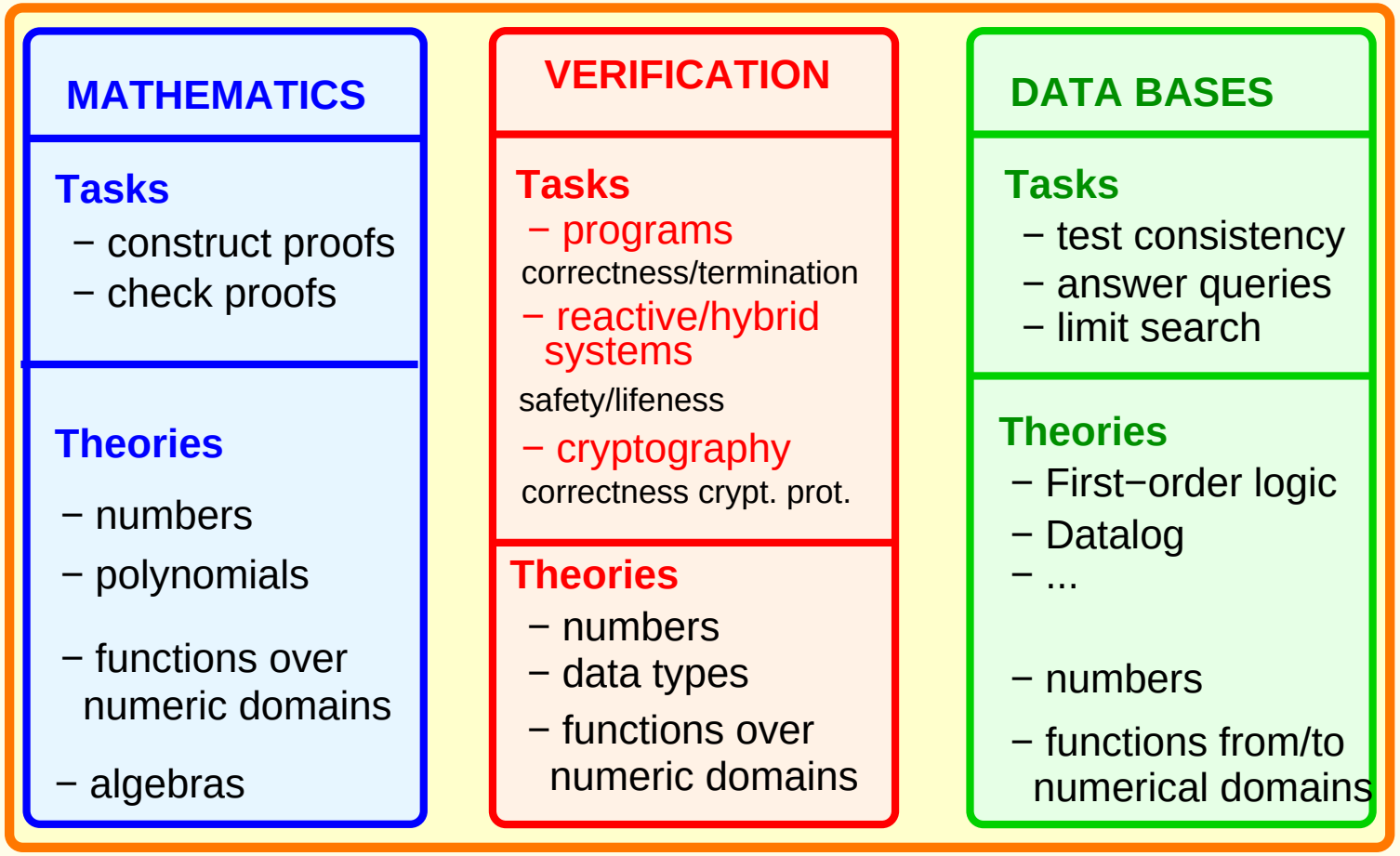
- non-disjoint sets of controlled trains
- synchronization for the control of common trains



- complex track topology



Examples of application domains



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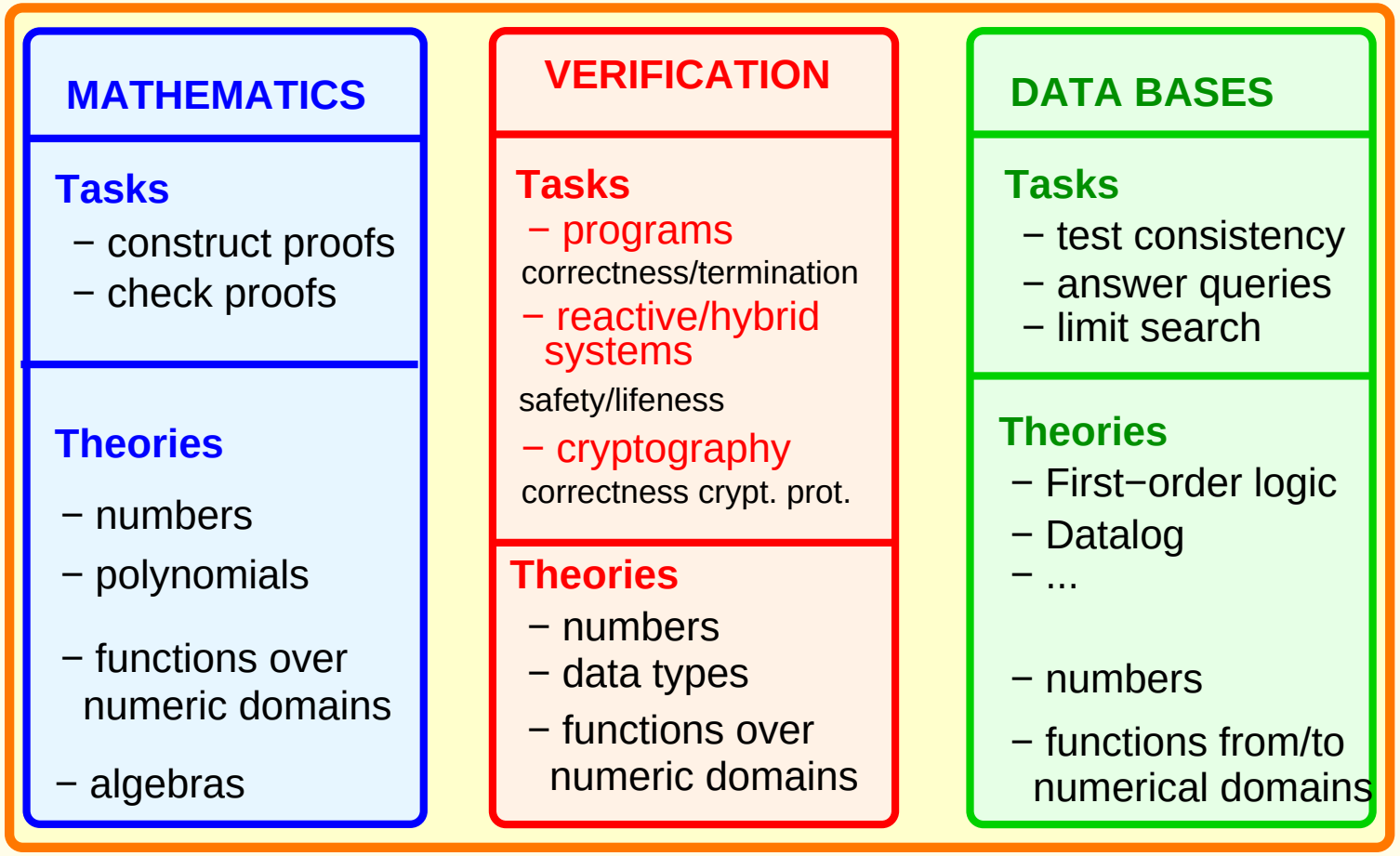

complex systems (MAS, reactive systems w. embedded software, databases)

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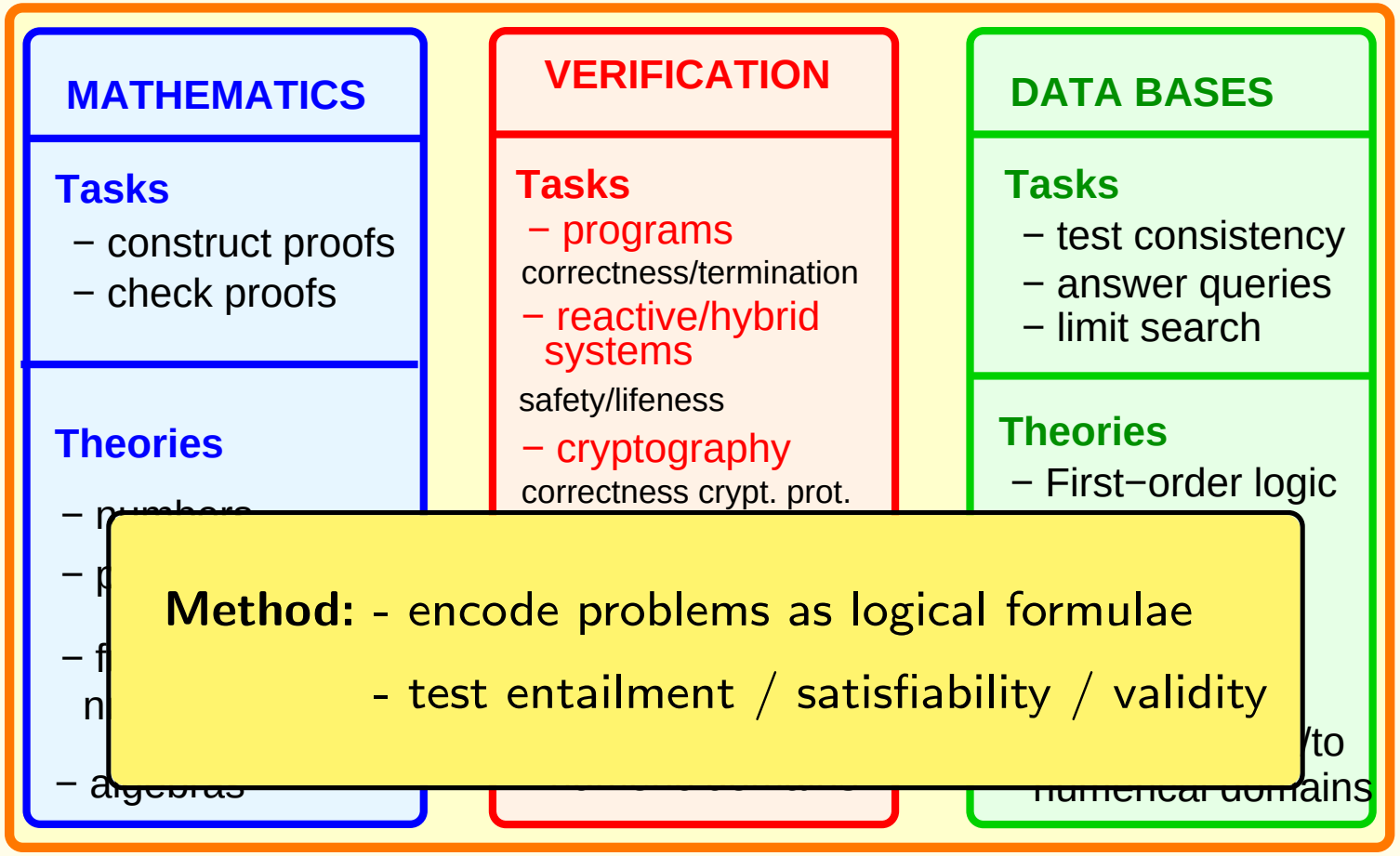

complex systems (MAS, reactive systems w. embedded software, databases)

Problems and goals

– 1^{st} order logic is undecidable: cannot build an 'all-purpose' program

\+ often fragments of theories occurring in applications are decidable

– theories do not occur alone: need to consider combinations of theories

\+ often provers for the component theories can be combined efficiently

Problems and goals

– 1^{st} order logic is undecidable: cannot build an 'all-purpose' program

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– theories do not occur alone: need to consider combinations of theories

\+ often provers for the component theories can be combined efficiently

Important:

Identify theories (and extensions/combinations thereof) which are decidable (with low complexity) and relevant in applications

Overview

Plan of the lecture:

- **Reasoning in standard theories**
- **Reasoning in theory extensions**
- **Reasoning in combinations of theories**

Important: identify decidable/tractable fragments

Reasoning about standard datatypes

- **Numbers** - natural numbers, integers, reals, rationals

- **Data structures**
  - theories of lists
  - theory of acyclic lists
  - theory of arrays
  - theories of sets, multisets

- **Algebraic theories**
  - (total/partial) orderings
  - lattices, semilattices
  - distributive lattices
  - Boolean algebras
  - groups, rings, fields, ...

Reasoning in theory extensions

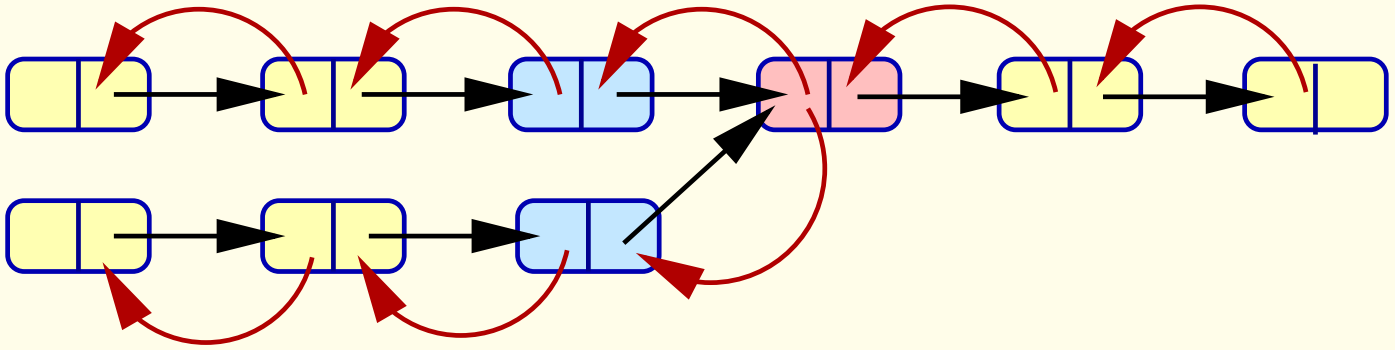
- **Numbers** - integers, reals, rationals

- **Data structures**
  - theories of lists of integers, reals, ...
  - theory of acyclic lists of integers, reals, ...
  - theory of arrays of integers, reals, ...
  - theories of sets of integers, reals, ...

  + functions (free, rec. def.) e.g : length, card

- **Algebraic theories**
  - (total/partial) orderings with monotone functions
  - lattices, semilattices with operators
  - distributive lattices with operators
  - Boolean algebras with operators
  - fields with operators

Example: A theory of doubly-linked lists

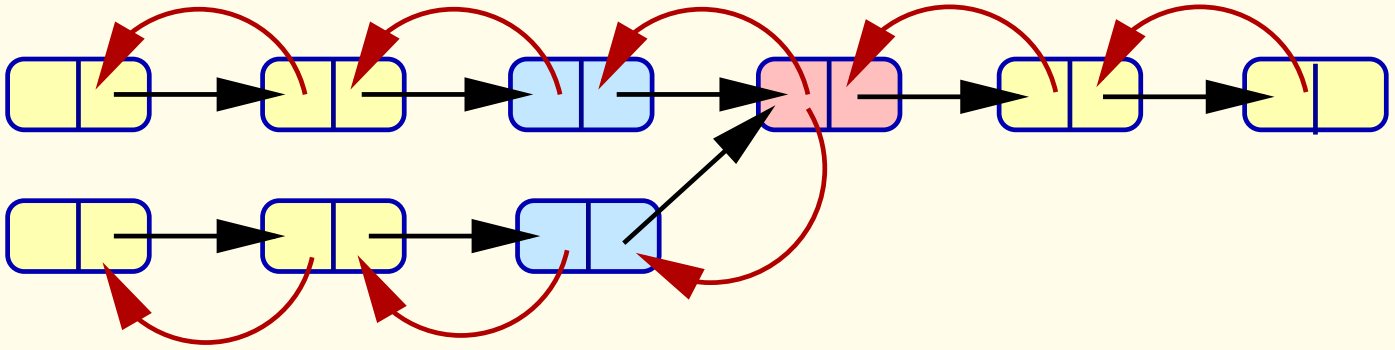


$$\forall p\ (p \neq \text{null} \wedge p.\text{next} \neq \text{null} \ \rightarrow \ p.\text{next.prev} = p)$$

$$\forall p\ (p \neq \text{null} \wedge p.\text{prev} \neq \text{null} \ \rightarrow \ p.\text{prev.next} = p)$$

$$\wedge \quad c{\neq}\text{null} \wedge c.\text{next}{\neq}\text{null} \wedge d{\neq}\text{null} \wedge d.\text{next}{\neq}\text{null} \wedge c.\text{next}{=}d.\text{next} \wedge c \neq d \quad \models \quad \bot$$

Example: A theory of doubly-linked lists



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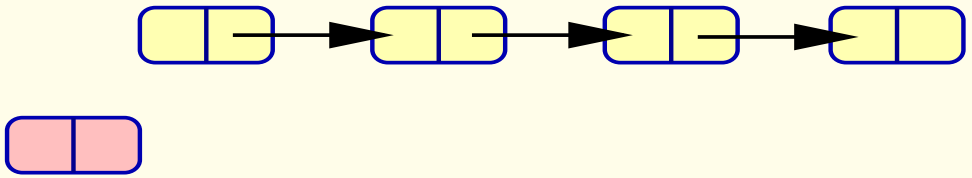
Scalar fields + additional axioms:

$$\forall p\ (p \neq \text{null} \wedge p.\text{next} \neq \text{null} \ \rightarrow \ p.\text{info} \leq p.\text{next.info})$$

Example: List insertion



Initially list is sorted: $p.\text{next} \neq \text{null} \rightarrow p.\text{info} \geq p.\text{next.info}$

$c.\text{info} = x, c.\text{next} = \text{null}$

for all $p \neq c$ **do**

if $p.\text{info} \leq x$ **then if** $\text{First}(p)$ **then** $c.\text{next}' = p$, $\text{First}'(c)$, $\neg\text{First}'(p)$ **endif**; $p.\text{next}' = p.\text{next}$

$p.\text{info} > x$ **then** **case** $p.\text{next} = \text{null}$ **then** $p.\text{next}' := c, c.\text{next}' = \text{null}$

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Verification task: After insertion list remains sorted

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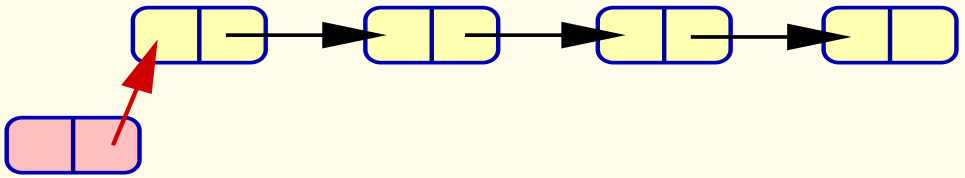

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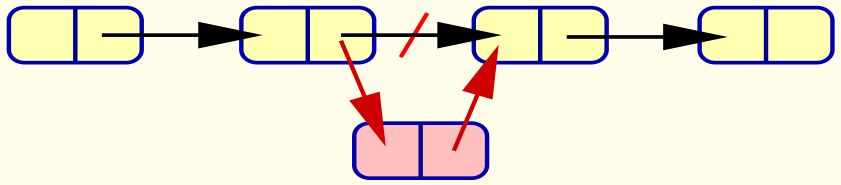

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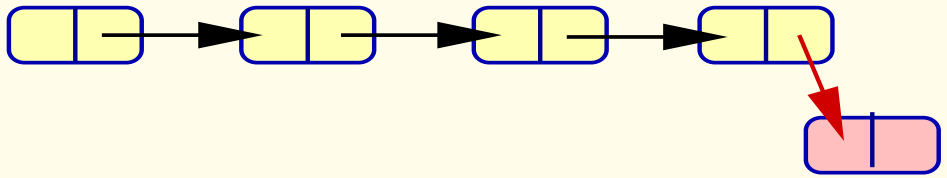

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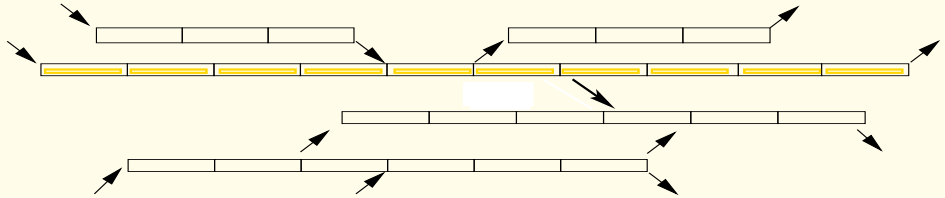
Applications

- Program verification
- More general verification problems

Model of a train controller

- **Complex track topologies** [Faber, Ihlemann, Jacobs, VS, IFM 2010]



Assumptions:

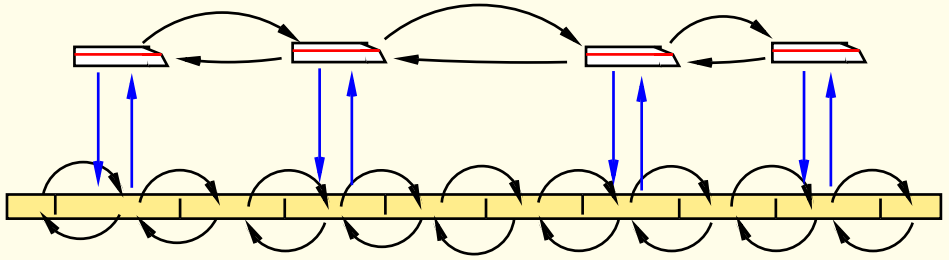
- No cycles
- in-degree (out-degree) of associated graph at most 2.

Data structures:

- 2-sorted pointers

  p_1: trains

  p_2: segments

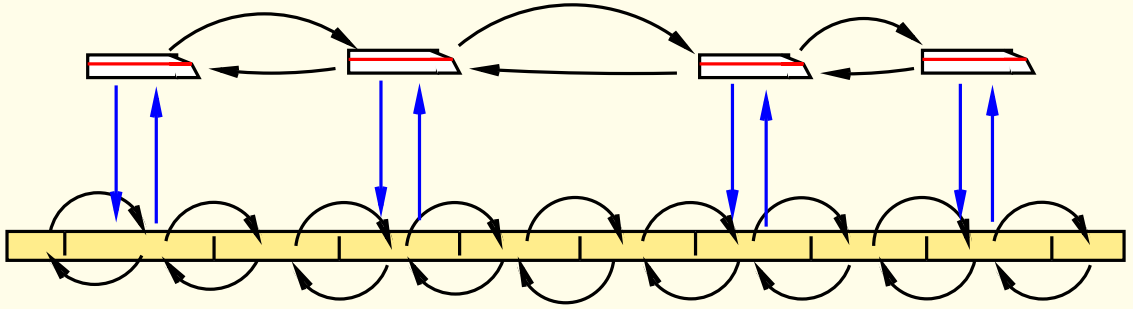


- scalar fields

  $\text{speed}: p_1 \to \mathbb{R}, \quad \text{id}_t : p_1 \to \mathbb{Z}$

  $\text{MaxSpeed}: p_2 \to \mathbb{R}, \quad \text{MaxBrakingDistance}: p_2 \to \mathbb{R}, \quad \text{id}_s : p_1 \to \mathbb{Z}$

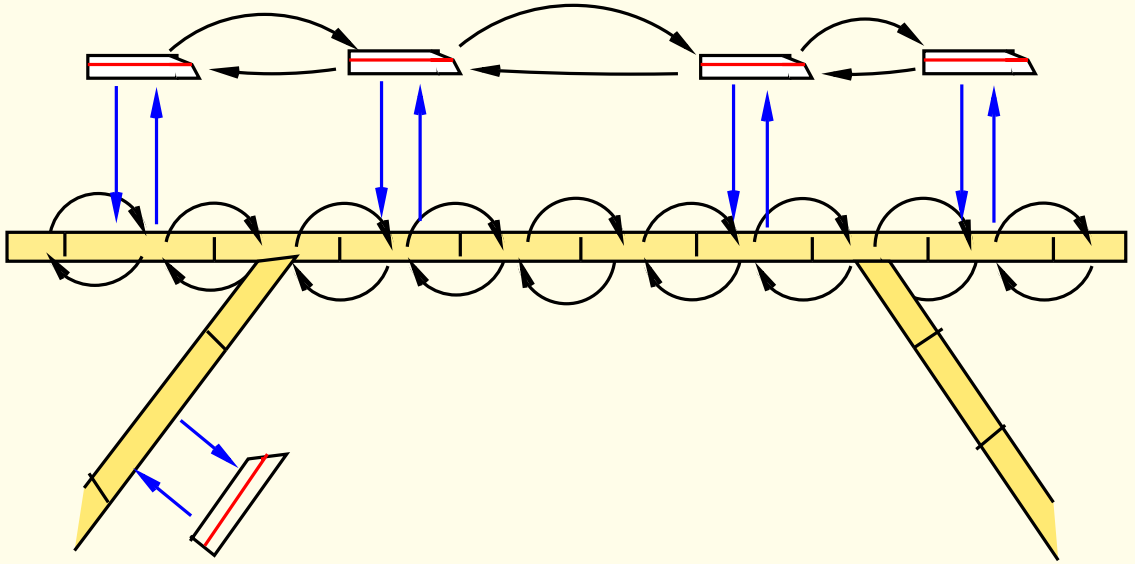
Incoming and outgoing trains



Example 1: Speed Update

$$\begin{aligned}
&\mathrm{pos}(t) < \mathrm{length}(\mathrm{segm}(t)) - d \;\rightarrow\; 0 \leq \mathrm{spd}'(t) \leq \mathrm{lmax}(\mathrm{segm}(t)) \\
&\mathrm{pos}(t) \geq \mathrm{length}(\mathrm{segm}(t)) - d \;\wedge\; \mathrm{alloc}(\mathrm{next}_s(\mathrm{segm}(t))) = \mathrm{tid}(t) \\
&\qquad\qquad \rightarrow\; 0 \leq \mathrm{spd}'(t) \leq \min(\mathrm{lmax}(\mathrm{segm}(t)), \mathrm{lmax}(\mathrm{next}_s(\mathrm{segm}(t)))) \\
&\mathrm{pos}(t) \geq \mathrm{length}(\mathrm{segm}(t)) - d \;\wedge\; \mathrm{alloc}(\mathrm{next}_s(\mathrm{segm}(t))) \neq \mathrm{tid}(t) \\
&\qquad\qquad \rightarrow\; \mathrm{spd}'(t) = \max(\mathrm{spd}(t) - \mathrm{decmax}, 0)
\end{aligned}$$

Incoming and outgoing trains



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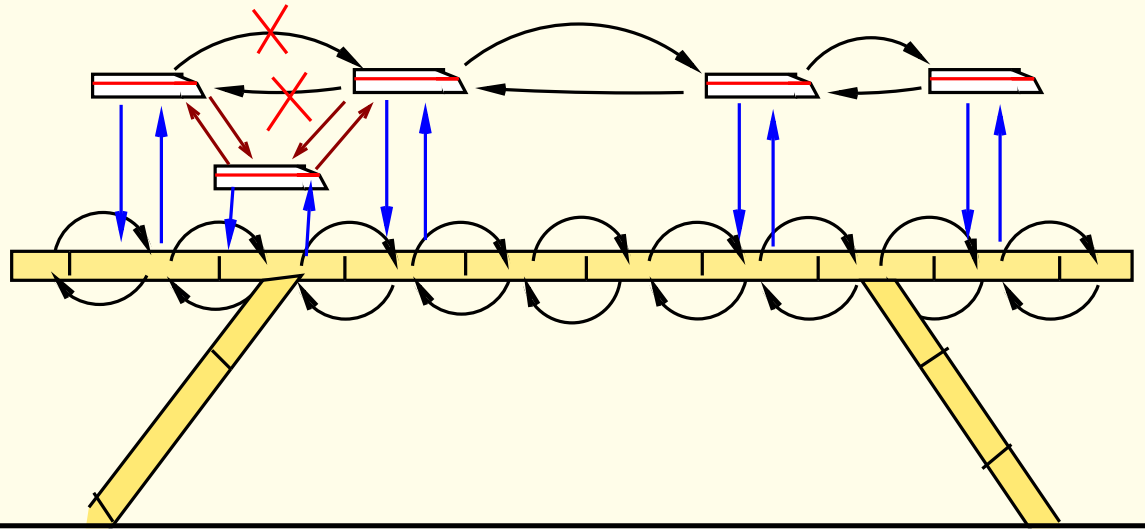

Example 2: Enter Update (also updates for segm', spd', pos', train')

Assume: $s_1 \neq \text{null}_s$, $t_1 \neq \text{null}_t$, $\text{train}(s) \neq t_1$, $\text{alloc}(s_1) = \text{idt}(t_1)$

$t{\neq}t_1, \text{ids}(\text{segm}(t)){<}\text{ids}(s_1), \text{next}_t(t){=}\text{null}_t, \text{alloc}(s_1){=}\text{tid}(t_1) \rightarrow \text{next}'(t){=}t_1 \wedge \text{next}'(t_1){=}\text{null}_t$

$t{\neq}t_1, \text{ids}(\text{segm}(t)){<}\text{ids}(s_1), \text{alloc}(s_1){=}\text{tid}(t_1), \text{next}_t(t){\neq}\text{null}_t, \text{ids}(\text{segm}(\text{next}_t(t))){\leq}\text{ids}(s_1)$

$\rightarrow \text{next}'(t){=}\text{next}_t(t)$

...

$t{\neq}t_1, \text{ids}(\text{segm}(t)){\geq}\text{ids}(s_1) \rightarrow \text{next}'(t){=}\text{next}_t(t)$

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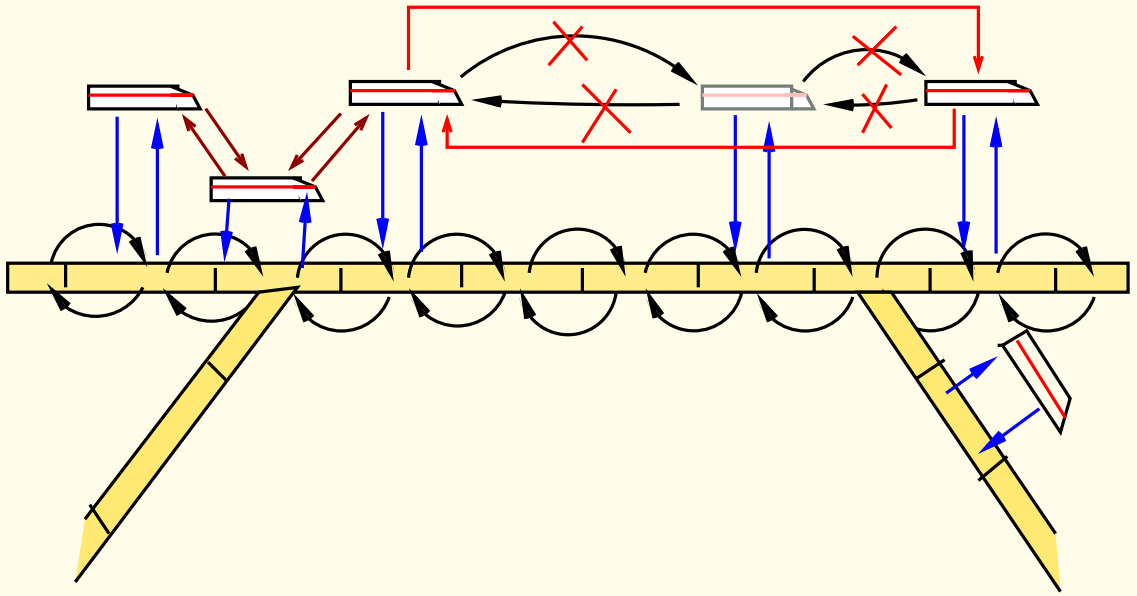

Safety property

Safety property we want to prove: no two trains ever occupy the same track segment:

$$(\text{Safe}) := \forall t_1, t_2 \;\; \text{segm}(t_1) = \text{segm}(t2) \rightarrow t_1 = t_2$$

We need to be able to efficiently reason in combinations of theories.

Example 2

Example: Does BubbleSort return a sorted array?

```
int [] BubbleSort(int[] a) {
    int i, j, t;
    for (i := |a| − 1; i > 0; i := i − 1) {
       for (j := 0; j < i; j := j + 1) {
          if (a[j] > a[j + 1]){t := a[j];
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```

Example 2

$-1 \leq i < |a| \wedge$
partitioned$(a, 0, i, i+1, |a|-1) \wedge$
sorted$(a, i, |a|-1)$

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Generate verification conditions and prove that they are valid

Predicates:

- sorted(a, l, u): $\forall i, j (l \leq i \leq j \leq u \rightarrow a[i] \leq a[j])$
- partitioned(a, l_1, u_1, l_2, u_2): $\forall i, j (l_1 \leq i \leq u_1 \leq l_2 \leq j \leq u_2 \rightarrow a[i] \leq a[j])$

Example 2

$-1 \leq i < |a| \wedge$ $\boxed{C_1(a)}$

$\text{partitioned}(a, 0, i, i+1, |a|-1) \wedge$

$\text{sorted}(a, i, |a|-1)$

$-1 \leq i < |a| \wedge 0 \leq j \leq i \wedge$ $\boxed{C_2(a)}$

$\text{partitioned}(a, 0, i, i+1, |a|-1) \wedge$

$\text{sorted}(a, i, |a|-1)$

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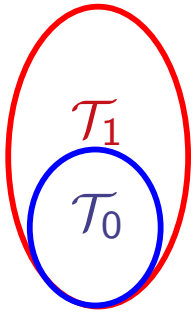
Generate verification conditions and prove that they are valid

$C_2(a) \wedge \text{Update}(a, a') \rightarrow C_2(a')$

Motivation

Modular (i.e. black-box) composition of decision procedures is highly desirable – for saving time and resources.

Idea



Hierarchic Reasoning

$\mathcal{T}_1$: Σ_1-theory; $\mathcal{T}_0 \subseteq \mathcal{T}_1$ $\Sigma_0 \subset \Sigma_1$

$\mathcal{T}_0$: Σ_0-theory.

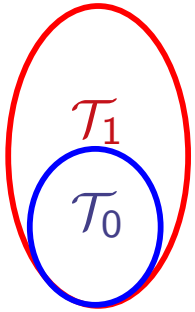
Example:

$f : \mathbb{R} \to \mathbb{R}$ mon.

$\mathbb{R}$

Can we use a prover for $\mathcal{T}_0$ as a blackbox to prove theorems in $\mathcal{T}_1$?

Idea



Hierarchic Reasoning

$\mathcal{T}_1$: Σ_1-theory; $\mathcal{T}_0 \subseteq \mathcal{T}_1$ $\Sigma_0 \subset \Sigma_1$

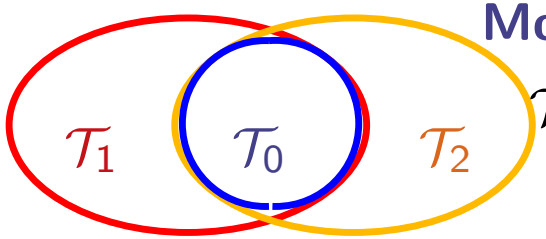
$\mathcal{T}_0$: Σ_0-theory.

Example:

$f : \mathbb{R} \to \mathbb{R}$ mon.

$\mathbb{R}$

Can we use a prover for $\mathcal{T}_0$ as a blackbox to prove theorems in $\mathcal{T}_1$?



Modular Reasoning

$\mathcal{T}_0$: Σ_0-theory.

$\mathcal{T}_i$: Σ_i-theory; $\mathcal{T}_0 \subseteq \mathcal{T}_i$ $\Sigma_0 \subseteq \Sigma_i$.

Example: **arrays**$(\mathbb{Z}, \mathbb{R})$

lists$(\mathbb{R})$ $\cup$ **arrays**$(\mathbb{Z}, \mathbb{R})$

Can we use provers for $\mathcal{T}_1, \mathcal{T}_2$ as blackboxes to prove theorems in $\mathcal{T}_1 \cup \mathcal{T}_2$?

Which information needs to be exchanged between the provers?

Structure

- **Reasoning in standard theories**

  Preliminaries: Logic, theories, models

  Decidable logical theories and theory fragments

  Tractability

- **Reasoning in combinations of theories**

  disjoint signature

  non-disjoint signature

- **Theory extensions**

- **Applications in Verification**

Reasoning in standard theories

- **Propositional logic and first-order logic** (reminder)
  - Syntax, Semantics, Entailment, Validity, Satisfiability
  - Theories, Models
  - Deduction in propositional logic (DPLL, resolution)
  - Deduction in first-order logic (the resolution principle)
- **Decidability and undecidability results**
  - Undecidability of FOL/Some decidable fragments
  - Deduction problems
- **Reasoning in numerical domains** (a crash course)

Reasoning in complex theories

- **Reasoning in combinations of theories**
  - combinations of theories with disjoint signatures
  - combinations of theories with non-disjoint signatures (main idea)
- **Theory extensions**
- **Applications**
  - Decision procedures for data structures:
    - arrays
    - pointer structures
    - sets (with cardinalities)
    - recursive data structures + recursive functions
  - Applications in verification

Literature

Logic:

Any book on logic (propositional and first-order logic)

- The slides of the logic lecture
- Schöning: Logik für Informatiker, Spektrum
- Fitting: First-Order Logic and Automated Theorem Proving, Springer

Literature

Decision procedures:

- Aaron Bradley and Zohar Manna: The Calculus of Computation: Decision Procedures with Applications to Verification. Springer, 2007.

- Daniel Kroening and Ofer Strichman: Decision Procedures: An algorithmic point of view. Springer, 2007.

Viorica Sofronie-Stokkermans: Reasoning in complex theories. Lecture notes/slides

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