Decision Procedures for Verification

Part 1. Propositional Logic (4)

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Last time

1.1 Syntax

- Language
  - propositional variables
  - logical symbols
    $\Rightarrow$ Boolean combinations
- Propositional Formulae

1.2 Semantics

- Valuations
- Truth value of a formula in a valuation
- Models, Validity, and Satisfiability
- Entailment and Equivalence

Canonical forms

- CNF and DNF
- Computing CNF/DNF by rewriting the formulae
- Structure-Preserving Translation for CNF
- Optimized translation using polarity

Decision Procedures for Satisfiability

- Simple Decision Procedures

  truth table method — Logik f. Informatiker, Discrete Algebraic Structures

- The Resolution Procedure (last time)

- The DPLL Procedure

1.7 The DPLL Procedure

Goal:

Given a propositional formula in CNF (or alternatively, a finite set N of clauses), check whether it is satisfiable (and optionally: output *one* solution, if it is satisfiable).

Satisfiability of Clause Sets

$\mathcal{A} \models N$ if and only if $\mathcal{A} \models C$ for all clauses C in N.

$\mathcal{A} \models C$ if and only if $\mathcal{A} \models L$ for some literal $L \in C$.

Partial Valuations

Since we will construct satisfying valuations incrementally,
we consider partial valuations
(that is, partial mappings $\mathcal{A} : \Pi \to \{0, 1\}$).

We start with an empty valuation and try to extend it
step by step to all variables occurring in N.

If $\mathcal{A}$ is a partial valuation, then literals and clauses can be
true, false, or undefined under $\mathcal{A}$.

A clause is true under $\mathcal{A}$ if one of its literals is true;
it is false (or "conflicting") if all its literals are false;
otherwise it is undefined (or "unresolved").

Unit Clauses

Observation:

Let $\mathcal{A}$ be a partial valuation. If the set N contains a clause C, such that all literals but one in C are false under $\mathcal{A}$, then the following properties are equivalent:

- there is a valuation that is a model of N and extends $\mathcal{A}$.
- there is a valuation that is a model of N and extends $\mathcal{A}$ and makes the remaining literal L of C true.

C is called a unit clause; L is called a unit literal.

Pure Literals

One more observation:

Let $\mathcal{A}$ be a partial valuation and P a variable that is undefined under $\mathcal{A}$. If P occurs only positively (or only negatively) in the unresolved clauses in N, then the following properties are equivalent:

- there is a valuation that is a model of N and extends $\mathcal{A}$.
- there is a valuation that is a model of N and extends $\mathcal{A}$ and assigns true (false) to P.

P is called a pure literal.

The Davis-Putnam-Logemann-Loveland Proc.

boolean DPLL(clause set N, partial valuation $\mathcal{A}$) {
    if (all clauses in N are true under $\mathcal{A}$) return true;
    elsif (some clause in N is false under $\mathcal{A}$) return false;
    elsif (N contains unit clause P) return DPLL(N, $\mathcal{A} \cup \{P \mapsto 1\}$);
    elsif (N contains unit clause $\neg P$) return DPLL(N, $\mathcal{A} \cup \{P \mapsto 0\}$);
    elsif (N contains pure literal P) return DPLL(N, $\mathcal{A} \cup \{P \mapsto 1\}$);
    elsif (N contains pure literal $\neg P$) return DPLL(N, $\mathcal{A} \cup \{P \mapsto 0\}$);
    else {
        let P be some undefined variable in N;
        if (DPLL(N, $\mathcal{A} \cup \{P \mapsto 0\}$)) return true;
        else return DPLL(N, $\mathcal{A} \cup \{P \mapsto 1\}$);
    }
}

The Davis-Putnam-Logemann-Loveland Proc.

Initially, DPLL is called with the clause set N and with an empty partial valuation $\mathcal{A}$.

The Davis-Putnam-Logemann-Loveland Proc.

In practice, there are several changes to the procedure:

The pure literal check is often omitted (it is too expensive).

The branching variable is not chosen randomly.

The algorithm is implemented iteratively;
the backtrack stack is managed explicitly
(it may be possible and useful to backtrack more than one level).

DPLL Iteratively

An iterative (and generalized) version:

```
status = preprocess();
if (status != UNKNOWN) return status;
while(1) {
    decide_next_branch();
    while(1) {
        status = deduce();
        if (status == CONFLICT) {
            blevel = analyze_conflict();
            if (blevel == 0) return UNSATISFIABLE;
            else backtrack(blevel); }
        else if (status == SATISFIABLE) return SATISFIABLE;
        else break;
    }
}
```

DPLL Iteratively

`preprocess()`

preprocess the input (as far as it is possible without branching);
return CONFLICT or SATISFIABLE or UNKNOWN.

`decide_next_branch()`

choose the right undefined variable to branch;
decide whether to set it to 0 or 1;
increase the backtrack level.

DPLL Iteratively

`deduce()`

make further assignments to variables (e.g., using the unit clause rule) until a satisfying assignment is found, or until a conflict is found, or until branching becomes necessary; return CONFLICT or SATISFIABLE or UNKNOWN.

DPLL Iteratively

`analyze_conflict()`

check where to backtrack.

`backtrack(blevel)`

backtrack to `blevel`;

flip the branching variable on that level;

undo the variable assignments in between.

Branching Heuristics

Choosing the right undefined variable to branch is important for efficiency, but the branching heuristics may be expensive itself.

State of the art: use branching heuristics that need not be recomputed too frequently.

In general: choose variables that occur frequently.

The Deduction Algorithm

For applying the unit rule, we need to know the number of literals in a clause that are not false.

Maintaining this number is expensive, however.

The Deduction Algorithm

Better approach: "Two watched literals":

In each clause, select two (currently undefined) "watched" literals.

For each variable P, keep a list of all clauses in which P is watched and a list of all clauses in which $\neg P$ is watched.

If an undefined variable is set to 0 (or to 1), check all clauses in which P (or $\neg P$) is watched and watch another literal (that is true or undefined) in this clause if possible.

Watched literal information need not be restored upon backtracking.

Conflict Analysis and Learning

Goal: Reuse information that is obtained in one branch in further branches.

Method: Learning:

> If a conflicting clause is found, use the resolution rule to derive a new clause and add it to the current set of clauses.

Problem: This may produce a large number of new clauses; therefore it may become necessary to delete some of them afterwards to save space.

Backjumping

Related technique:

non-chronological backtracking ("backjumping"):

> If a conflict is independent of some earlier branch, try to skip that over that backtrack level.

Restart

Runtimes of DPLL-style procedures depend extremely on the choice of branching variables.

If no solution is found within a certain time limit, it can be useful to restart from scratch with another choice of branchings (but learned clauses may be kept).

A succinct formulation

State: $M||F$,

where:

- M partial assignment (sequence of literals),

  some literals are annotated (L^d: decision literal)

- F clause set.

A succinct formulation

UnitPropagation

$M||F, C \vee L \Rightarrow M, L||F, C \vee L$ if $M \models \neg C$, and L undef. in M

Decide

$M||F \Rightarrow M, L^d||F$ if L or $\neg L$ occurs in F, L undef. in M

Fail

$M||F, C \Rightarrow \text{Fail}$ if $M \models \neg C$, M contains no decision literals

Backjump

$M, L^d, N||F \Rightarrow M, L'||F$ if
- there is some clause $C \vee L'$ s.t.:
- $F \models C \vee L', M \models \neg C$,
- L' undefined in M
- L' or $\neg L'$ occurs in F.

Backjump

Backjump

$$M, L^d, N||F \Rightarrow M, L'||F \qquad \text{if} \begin{cases} \text{there is some clause } C \vee L' \text{ s.t.:} \\ F \models C \vee L', M \models \neg C, \\ L' \text{ undefined in } M \\ L' \text{ or } \neg L' \text{ occurs in } F. \end{cases}$$

We will see later that the Backjump rule is always applicable, if the list of literals ML^dN contains at least one decision literal and some clause in N is false under M. There are many possible backjump clauses. One candidate: $\neg L_1 \vee \cdots \vee \neg L_n$, where the L_i are all the decision literals in ML^dN. (But usually there are better choices.)

Example

Assignment:	Clause set:	
$\emptyset$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)

Example

Assignment:	Clause set:	
$\emptyset$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
P_1^d	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)

Example

Assignment:	Clause set:	
$\emptyset$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
P_1^d	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)

Example

Assignment:	Clause set:	
$\emptyset$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
P_1^d	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
$P_1^d P_2 P_3^d$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)

Example

Assignment:	Clause set:	
$\emptyset$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
P_1^d	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
$P_1^d P_2 P_3^d$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2 P_3^d P_4$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)

Example

Assignment:	Clause set:	
$\emptyset$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
P_1^d	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
$P_1^d P_2 P_3^d$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2 P_3^d P_4$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
$P_1^d P_2 P_3^d P_4 P_5^d$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)

Example

Assignment:	Clause set:	
$\emptyset$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
P_1^d	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
$P_1^d P_2 P_3^d$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2 P_3^d P_4$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
$P_1^d P_2 P_3^d P_4 P_5^d$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2 P_3^d P_4 P_5^d \neg P_6$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Backtrack)

Example

Assignment:	Clause set:	
$\emptyset$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
P_1^d	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
$P_1^d P_2 P_3^d$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2 P_3^d P_4$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
$P_1^d P_2 P_3^d P_4 P_5^d$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2 P_3^d P_4 P_5^d \neg P_6$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Backtrack)
$P_1^d P_2 P_3^d P_4 \neg P_5$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	...

DPLL with learning

The DPLL system with learning consists of the four transition rules of the Basic DPLL system, plus the following two additional rules:

Learn

$M||F \Rightarrow M||F, C$ if all atoms of C occur in F and $F \models C$

Forget

$M||F, C \Rightarrow M||F$ if $F \models C$

In these two rules, the clause C is said to be learned and forgotten, respectively.

Example

Assignment:	Clause set:	
$\emptyset$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
P_1^d	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
$P_1^d P_2 P_3^d$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2 P_3^d P_4$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
$P_1^d P_2 P_3^d P_4 P_5^d$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2 P_3^d P_4 P_5^d \neg P_6$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Backtrack)

Backjumping

Assignment:	Clause set:	
$\emptyset$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
P_1^d	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
$P_1^d P_2 P_3^d$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2 P_3^d P_4$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
$P_1^d P_2 P_3^d P_4 P_5^d$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2 P_3^d P_4 P_5^d \neg P_6$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	

Conflict clause: $P_6 \vee \neg P_5 \vee \neg P_2$

Backjumping

Assignment:	Clause set:	
$\emptyset$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
P_1^d	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
$P_1^d P_2 P_3^d$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2 P_3^d P_4$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
$P_1^d P_2 P_3^d P_4 P_5^d$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2 P_3^d P_4 P_5^d \neg P_6$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	

Conflict clause: $P_6 \vee \neg P_5 \vee \neg P_2$

Decisions ${P_1}^d$, ${P_3}^d$, and ${P_5}^d$ are the cause; cannot be taken at the same time.

Possible backjump clause: $\neg P_1 \vee \neg P_3 \vee \neg P_5$. Can we do better?

Backjumping

Assignment:	Clause set:	
$\emptyset$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
P_1^d	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
$P_1^d P_2 P_3^d$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2 P_3^d P_4$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
$P_1^d P_2 P_3^d P_4 P_5^d$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2 P_3^d P_4 P_5^d \neg P_6$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	

Conflict clause: $P_6 \vee \neg P_5 \vee \neg P_2$

Decisions ${P_1}^d$ and ${P_5}^d$ are the cause, but not P_3^d

Decisions ${P_1}^d$ and ${P_5}^d$ cannot be taken at the same time $\mapsto$ derive new clause $\neg P_1 \vee \neg P_5$; use for backjump.

Backjumping

Assignment:	Clause set:	
$\emptyset$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
P_1^d	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
$P_1^d P_2 P_3^d$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2 P_3^d P_4$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
$P_1^d P_2 P_3^d P_4 P_5^d$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (UnitProp)
$P_1^d P_2 P_3^d P_4 P_5^d \neg P_6$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Backjump)
$P_1^d P_2 \neg P_5$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	$\Rightarrow$ (Decide)
$P_1^d P_2 \neg P_5 P_4^d$	$\|\|\neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$	

Conflict clause: $P_6 \vee \neg P_5 \vee \neg P_2$

Decisions ${P_1}^d$ and ${P_5}^d$ are the cause, but not P_3^d

Decisions ${P_1}^d$ and ${P_5}^d$ cannot be taken at the same time $\mapsto$ derive new clause $\neg P_1 \vee \neg P_5$; use for backjump.

Backjump

Backjump

$$M, L^d, N||F \Rightarrow M, L'||F \qquad \text{if} \begin{cases} \text{there is some clause } C \vee L' \text{ s.t.:} \\ F \models C \vee L', M \models \neg C, \\ L' \text{ undefined in } M \\ L' \text{ or } \neg L' \text{ occurs in } F. \end{cases}$$

C conflict clause; $C \vee L'$ backjump clause

Backjump

Backjump

$$M, L^d, N || F, C \Rightarrow M, L' || F, C \qquad \text{if} \begin{cases} ML^d N \models \neg C \\ \text{There is some clause } C' \vee L' \text{ s.t.:} \\ F, C \models C' \vee L', M \models \neg C', \\ L' \text{ undefined in } M \\ L' \text{ or } \neg L' \text{ occurs in } F. \end{cases}$$

C conflict clause; $C' \vee L'$ backjump clause

$P_1^d P_2 P_3^d P_4 P_5^d \neg P_6 \quad || \neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2 \quad \Rightarrow$ (Backjump)

$P_1^d P_2 \neg P_5 \quad || \neg P_1 \vee P_2, \neg P_3 \vee P_4, \neg P_5 \vee \neg P_6, P_6 \vee \neg P_5 \vee \neg P_2$

Backjump clause: $C' \vee L'$
with $C' = \neg P_1$ and $L' = \neg P_5$ (Lemma, learned clause)

Determined e.g. using a "conflict graph".

Backjump

Suppose that we have reached a state $M||F$ such that some clause $C \in F$ (or following from F) is false under M.

Consequently, every literal of C is the complement of some literal in M.

(1) If every literal in C is the complement of a decision literal of M , then C is a backjump clause.

(2) Otherwise, $C = C' \vee \neg L$, such that L is a deduced literal.

For every deduced literal L, there is a (unit or backjump) clause $D \vee L$, such that $F \models D \vee L$ and D is false under M.

Then $F \models D \vee C'$ and $D \vee C'$ is also false under M.

($D \vee C'$ is a resolvent of $C' \vee \neg L$ and $D \vee L$.)

Backjump

By repeating this process, we will eventually obtain a clause that satisfies the requirements of a backjump clause (or the empty clause). Usually, one resolves the literals in the reverse order in which they were added to M and stops as soon as one obtains a clause in which all but one literal are complements of literals occurring in M before the last decision literal.

$\mapsto$ 1UIP (first unique implication point) strategy.

Further Information

The ideas described so far heve been implemented in the SAT checker Chaff.

Further information:
Lintao Zhang and Sharad Malik:
The Quest for Efficient Boolean Satisfiability Solvers,
Proc. CADE-18, LNAI 2392, pp. 295–312, Springer, 2002.

Applications of propositional logic

- A toy example (sudoku)
- Scheduling
- Verification

Sudoku

1							1	
2	4							
3		2						
4					5		4		7
5			8				3		
6			1		9				
7	3			4			2		
8		5		1					
9				8		6			

Idea: $p_{i,j}^{d}$ = true iff the value of square i, j is d

For example: $p_{3,5}^{8}$ = true

Sudoku

1							1	
2	4							
3		2						
4					5		4		7
5			8				3		
6			1		9				
7	3			4			2		
8		5		1					
9				8		6			

Coding SUDOKU by propositional clauses:

- Concrete values result in units: $p^d_{i,j}$.
- For every value, column we generate: $\neg p^d_{i,j} \vee \neg p^d_{i,k}$ (if $j \neq k$).
  Accordingly for all rows and 3×3 boxes.
- For every square we generate: $p^1_{i,j} \vee \ldots p^9_{i,j}$.
  For every two different values d, d', and every square we generate: $\neg p^d_{i,j} \vee \neg p^{d'}_{i,j}$.
- For every value d and every column we generate:
  $p^d_{i,1} \vee \ldots p^d_{i,9}$.
  Accordingly for all rows and 3×3 boxes.

Sudoku

1							1	
2	4							
3		2						
4					5		4		7
5			8				3		
6			1		9				
7	3			4			2		
8		5		1					
9				8		6			

Set of clauses satisfiable $\Leftrightarrow$ Sudoku has a solution

Let $\mathcal{A}$ be a satisfying assignment

$\mathcal{A}(p_{i,j}^k) = 1$ iff a k appears in line i, column j.

Scheduling

Example: A simple scheduling problem

In a school there are three teachers with the following specialization combinations:

Müller	Mathematics
Schmidt	German
Körner	Mathematics, German

	Group a	Group b
8:00– 8:50	Mathematics	German
9:00– 9:50	German	German
10:00–10:50	Math	Mathematics

Each teacher must teach at least two classes.

Scheduling

Müller	Mathematics
Schmidt	German
Körner	Mathematics, German

	Group a	Group b
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2) 9:00– 9:50	German	German
3)10:00–10:50	Math	Mathematics

Modeling:

Propositional variables: $P_{s,k,N,f}$ 'Teacher N teaches subject f in group k in time slot s'

Scheduling

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Modeling:

Propositional variables: $P_{s,k,N,f}$ 'Teacher N teaches subject f in group k in time slot s'

Rules: $(P_{1,a,M,m} \vee P_{1,a,K,m}) \wedge (P_{1,b,S,d} \vee P_{1,b,K,d})$

$(P_{2,a,S,d} \vee P_{2,a,K,d}) \wedge (P_{2,b,S,d} \vee P_{2,b,K,d})$

$(P_{3,a,M,m} \vee P_{3,a,K,m}) \wedge (P_{3,b,S,d} \vee P_{3,a,K,d})$

$\neg(P_{1,a,K,m} \wedge P_{1,b,K,d}) \wedge \neg(P_{2,a,K,d} \wedge P_{2,b,K,d}) \wedge \neg(P_{2,a,S,d} \wedge P_{2,b,S,d}) \wedge$

$\neg(P_{3,a,K,m} \wedge P_{3,b,K,m}) \wedge (P_{1,a,M,m} \wedge P_{1,b,M,m}) \ldots$

Program Verification

- Bounded model checking
- Model checking

- Invariant checking/generation

- Abstraction

Finite-state systems

- X finite set of variables, V finite set of possible values for the variables p^i_{xv} (in the i-th step x takes value v)
- Other propositional variables $q_k, k \in K$
- Transitions (variables change their value)

  $Tr(i, i+1) := \bigvee \left(\text{Cond}(p^i_{x_1 v_1^i}, \ldots, p^i_{x_n v_n^i}) \wedge \bigwedge_{j=1}^n p^{i+1}_{x_j v_j^{i+1}} \wedge \bigwedge_k q_k^{i+1} \right)$

  (where v_j^{i+1}, q_k^{i+1} suitably computed)

$F(p^k_{x_1, v_1^k}, \ldots, p^k_{x_n, v_n^k}, \ldots)$ property of assignments

Bounded model checking:

$$\bigwedge_{j=1}^n p^1_{x_j, v_j} \wedge \bigwedge q_k^1 \wedge Tr(1,2) \wedge \ldots \wedge Tr(k-1, k) \wedge \neg F(p^k_{x_1, v_1^k}, \ldots, p^k_{x_n, v_n^k}, \ldots)$$

Example

Question: Does BubbleSort return a sorted array?

```
int [] BubbleSort(int[] a) {
    int i, j, t;
    for (i := |a| − 1; i > 0; i := i − 1) {
        for (j := 0; j < i; j := j + 1) {
            if (a[j] > a[j + 1]){t := a[j];
                                 a[j] := a[j + 1];
                                 a[j + 1] := t};
}} return a}
```

Example

Question: Does BubbleSort return a sorted array?

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int [] BubbleSort(int[] a) {
    int i, j, t;
    for (i := |a| − 1; i > 0; i := i − 1) {
        for (j := 0; j < i; j := j + 1) {
            if (a[j] > a[j + 1]){t := a[j];
                                 a[j] := a[j + 1];
                                 a[j + 1] := t};
}} return a}
```

Simpler question:

$|a| = 3$; $a[0]{=}7, a[1]{=}9, a[2]{=}4$

does BubbleSort applied to this array return a sorted array?

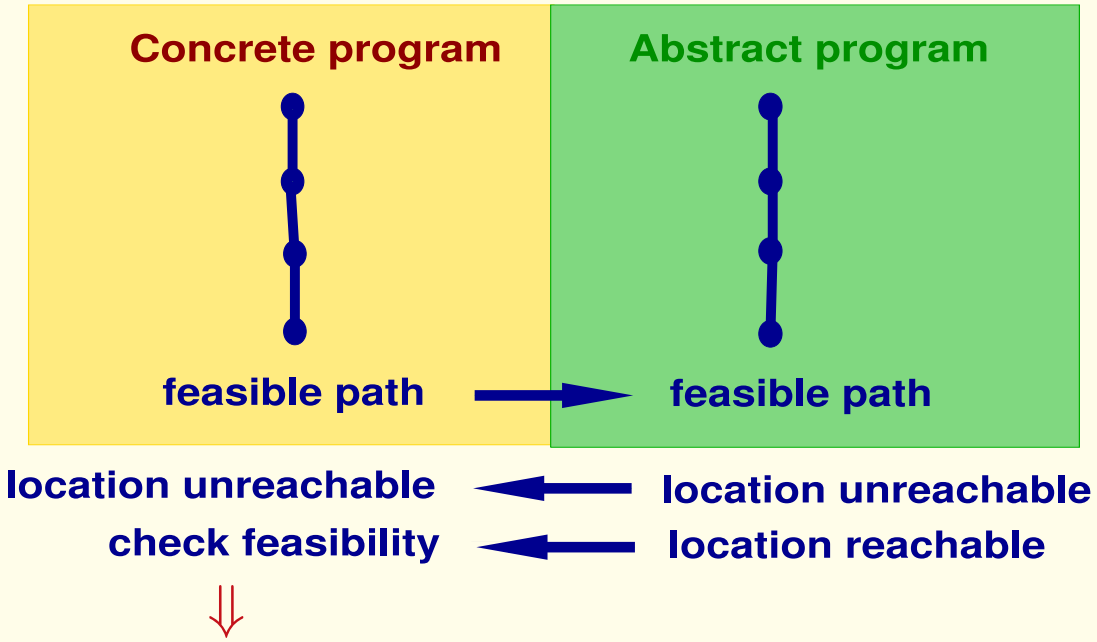
Encoding in propositional logic:

- p_{ij}^k (at step k, $a[i] = j$)
  Examples: $p_{07}^1, p_{19}^1, p_{24}^1$
- gt_{ij}^k (at step k, $a[i] > a[j]$)
  Examples: $gt_{10}^1, \neg gt_{01}^1, gt_{02}^1, \neg gt_{20}^1, \ldots$

Model updates with new propositional variables

(complicated; not very expressive)

Abstraction-Based Verification



conjunction of constraints: $\phi(1) \wedge Tr(1,2) \wedge \cdots \wedge Tr(n-1,n) \wedge \neg \mathsf{safe}(n)$

- satisfiable: feasible path
- unsatisfiable: refine abstract program s.t. the path is not feasible

Tools for SAT checking

`http://www.satcompetition.org/`

Examples of SAT solvers:

MiniSat: `http://minisat.se/`

MathSAT: `http://mathsat.fbk.eu/publications.html` (much more)

zChaff: `http://www.princeton.edu/ chaff/zchaff.html`

Example of use

Tools for SAT checking

Resolution-based theorem provers:

E: `http://www4.informatik.tu-muenchen.de/ schulz/E/E.html`

SPASS: `http://www.spass-prover.org/`

Vampire: `http://www.vprover.org/`

. . . full power for first-order logic (with equality)