Universität Koblenz-Landau

FB 4 Informatik

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Exercises for "Formal Specification and Verification"
Exercise sheet 4

Exercise 4.1:

Let Σ be a signature, $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2$ be sets of (closed) first-order Σ-formulae and $\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2$ be families of Σ-algebras. Prove (with the notations used in the lecture):

(1a)	$\mathcal{F} \subseteq \mathsf{Th}(\mathsf{Mod}(\mathcal{F}))$	(1b)	$\mathcal{M} \subseteq \mathsf{Mod}(\mathsf{Th}(\mathcal{M}))$
(2a)	If $\mathcal{F}_1 \subseteq \mathcal{F}_2$ then $\mathsf{Mod}(\mathcal{F}_2) \subseteq \mathsf{Mod}(\mathcal{F}_1)$	(2b)	If $\mathcal{M}_1 \subseteq \mathcal{M}_2$ then $\mathsf{Th}(\mathcal{M}_2) \subseteq \mathsf{Th}(\mathcal{M}_1)$
(3a)	$\mathsf{Mod}(\mathsf{Th}(\mathsf{Mod}(\mathcal{F}))) = \mathsf{Mod}(\mathcal{F})$	(3b)	$\mathsf{Th}(\mathsf{Mod}(\mathsf{Th}(\mathcal{M}))) = \mathsf{Th}(\mathcal{M})$

Hint: For proving each of the inclusions in (3a) and (3b) one can for instance use (1a,b) alone, resp. in combination with (2a,b).

Definitions and notations:

Let $\Sigma{=}(\Omega, \Pi)$ be a signature and $\mathcal{A}{=}(U, \{f_{\mathcal{A}}{:}U^n{\to}U\}_{f/n\in\Omega}, \{p_{\mathcal{A}}{:}U^m{\to}\{0,1\}_{p/m\in\Pi}\})$ be a Σ-structure.

- An equivalence relation $\sim\subseteq U{\times}U$ is a *congruence relation*[1] if it is compatible with the operations and predicates, i.e. for every $f/n{\in}\Omega$ and $p/m{\in}\Pi$:
  $$\forall x_1, \dots, x_n, y_1, \dots, y_n \in U,\ (x_1 \sim y_1 \wedge \dots \wedge x_n \sim y_n \rightarrow f_{\mathcal{A}}(x_1, \dots, x_n) \sim f_{\mathcal{A}}(y_1, \dots, y_n))$$
  $$\forall x_1, \dots, x_m, y_1, \dots, y_m \in U,\ (x_1 \sim y_1 \wedge \dots \wedge x_m \sim y_m \rightarrow p_{\mathcal{A}}(x_1, \dots, x_m) = p_{\mathcal{A}}(y_1, \dots, y_m)).$$
- *The quotient structure* is $\hat{\mathcal{A}}{=}\mathcal{A}/{\sim}{=}(\hat{U}, \{f_{\hat{\mathcal{A}}}{:}\hat{U}^n{\to}\hat{U}\}_{f/n\in\Omega}, \{p_{\hat{\mathcal{A}}}{:}\hat{U}^m{\to}\{0,1\}_{p/m\in\Pi})$, where:
  - $\hat{U} = U/{\sim} = \{[x] \mid x \in U\}$, where $[x] = \{y \in U \mid x \sim y\}$
  - $f_{\hat{\mathcal{A}}}([x_1], \dots, [x_n]) = [f(x_1, \dots, x_n)]$ for every $f/n \in \Omega$
  - $p_{\hat{\mathcal{A}}}([x_1], \dots, [x_m]) = p(x_1, \dots, x_n)$ for every $p/m \in \Pi$.
- A *term Σ-structure* (or a *Herbrand interpretation over* Σ) is a Σ-structure $\mathcal{A}$ having as universe the set T_Σ of ground terms and the operations defined by $f_{\mathcal{A}}(t_1, \dots, t_n) = f(t_1, \dots, t_n)$ and arbitrarily defined predicates.
- If $\Pi = \emptyset$ (i.e. $\approx$ is the unique predicate) then there is only one term Σ-structure (Herbrand interpretation), which we will denote with $\mathcal{T}_\Sigma$.

Exercise 4.2:

$\Sigma = (\Omega, \Pi)$ with $\Omega = \{b/0, f/1\}$ and $\Pi = \{p/1\}$.

(1) How many different Herbrand interpretations over Σ exist? Explain briefly.

(2) Consider the formula $F := p(f(f(b))) \wedge \forall x\, (p(x) \rightarrow p(f(x)))$. How many different Herbrand models over Σ does the formula F have? Explain briefly.

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[1] In the many-sorted case the definitions are similar, with the difference that $T_\Sigma = \{T^s_\Sigma\}_{s\in S}$, where T^s_Σ is the set of all terms of sort s, and a congruence relation $\sim$ consists of a family of equivalence relations $\{\sim_s\subseteq U_s \times U_s\}_{s\in S}$ which is compatible with the operations and predicates (similar definition, but the sorts of the variables x_i, y_i correspond to the arity of f; for comparing two terms of sort s the predicate $\sim_s$ is used).

(3) Every Herbrand interpretation which is a model of F is also a model of $G := \forall x\, p(f(f(x)))$. Give an example of an algebra that is a model of F but not of G.

(4) Let $\mathcal{A}$ be a Herbrand interpretation over Σ and let $\sim$ be the binary relation on T_Σ defined by:

$$t_1 \sim t_2 \text{ iff } \forall x\,(f(f(f(x))) = x) \models t_1 \approx t_2.$$

- Is $\sim$ a congruence relation on $\mathcal{A}$?
- Describe the quotient structure $\mathcal{A}/{\sim}$.
- Describe the class $\{\mathcal{A}/{\sim} \mid \mathcal{A} \text{ Herbrand interpretation over } \Sigma\}$.

Exercise 4.3:
Consider the following specification of binary trees (in a variant of the CASL syntax)

spec $\mathsf{BinTree} =$
sort $\mathsf{elem}, \mathsf{tree}$
operations $a :\to \mathsf{elem}$
$\mathsf{empty} :\to \mathsf{tree}$
$\mathsf{leaf} : \mathsf{elem} \to \mathsf{tree}$
$\mathsf{make} : \mathsf{tree}, \mathsf{tree} \to \mathsf{tree}$
$\mathsf{right} : \mathsf{tree} \to \mathsf{tree}$
$\mathsf{left} : \mathsf{tree} \to \mathsf{tree}$
Axioms: $\forall x_1, x_2 : \mathsf{tree}, \forall e : \mathsf{elem}$:
- $\mathsf{right}(\mathsf{empty}) \approx \mathsf{empty}$
- $\mathsf{right}(\mathsf{leaf}(e)) \approx \mathsf{empty}$
- $\mathsf{left}(\mathsf{empty}) \approx \mathsf{empty}$
- $\mathsf{left}(\mathsf{leaf}(e)) \approx \mathsf{empty}$
- $\mathsf{left}(\mathsf{make}(x_1, x_2)) \approx x_1$
- $\mathsf{right}(\mathsf{make}(x_1, x_2)) \approx x_2$

(1) Let $\mathcal{F}$ be the set of axioms in the specification above. Which of the following hold?

(1a) $\mathcal{F} \models \mathsf{left}(\mathsf{make}(\mathsf{empty}, \mathsf{empty})) \approx \mathsf{empty}$

(1b) $\mathcal{F} \models \mathsf{make}(x_1, x_2) = \mathsf{empty}$

(1c) $\mathcal{F} \models (x_2 \approx \mathsf{empty} \wedge x_3 \approx \mathsf{make}(x_1, \mathsf{empty})) \to \mathsf{make}(\mathsf{left}(\mathsf{make}(x_1, x_2)), \mathsf{right}(\mathsf{leaf}(e)) \approx x_3$

(1d) $\mathcal{F} \models \mathsf{make}(x_1, \mathsf{make}(x_2, x_3)) = x_2$

(2) Let $\sim$ be defined on T_Σ by:

$$t_1 \sim t_2 \text{ iff } \mathcal{F} \models t_1 \approx t_2.$$

Describe the quotient algebra $T_\Sigma/{\sim}$.

(3) Let $\sim'$ be defined on T_Σ by

$$t_1 \sim' t_2 \text{ iff } (\mathcal{F} \cup \{\forall x\, \mathsf{left}(x) \approx \mathsf{right}(x)\} \models t_1 \approx t_2).$$

Describe the quotient algebra $T_\Sigma/{\sim'}$.

Supplementary exercise

Exercise 4.4:
Let $\Sigma = (\Omega, \Pi)$ be a signature.

(1) Let $\mathcal{M}$ be a class of Σ-algebras and let $\sim \subseteq T_\Sigma \times T_\Sigma$ be defined by

$$t_1 \sim t_2 \text{ iff (for every } \mathcal{A} \in \mathcal{M} \text{ we have } \mathcal{A} \models t_1 \approx t_2).$$

(1a) Show that $\sim$ is a congruence relation on any Herbrand interpretation $\mathcal{A}$.

(1b) Assume that $\Pi = \emptyset$. Then there exists only one Herbrand interpretation, namely $\mathcal{T}_\Sigma$. Show that $\mathcal{T}_\Sigma$ has the property that for every $\mathcal{A} \in \mathcal{M}$ there exists a unique map $h : T_\Sigma \to U_\mathcal{A}$ such that for every $f/n \in \Omega$, $t_1, \dots, t_n \in T_\Sigma$ we have

$$h(f_{\mathcal{T}_\Sigma}(t_1, \dots, t_n)) = f_\mathcal{A}(h(t_1), \dots, h(t_n)).$$

(2) Let $\mathcal{F}$ be a set of (closed) formulae, and let $\sim \subseteq T_\Sigma \times T_\Sigma$ be defined by

$$t_1 \sim t_2 \text{ iff } \mathcal{F} \models t_1 \approx t_2.$$

Show that $\sim$ is an congruence relation on any Herbrand interpretation $\mathcal{A}$.

Please submit your solution until Wednesday, June 13, 2012 at 11:00.

Submission possibilities:

- By e-mail to `sofronie@uni-koblenz.de` with the keyword "Homework FSW" in the subject.
- Hand it in to me (Room B225) or drop it in the box in front of Room B224.