Formal Specification and Verification

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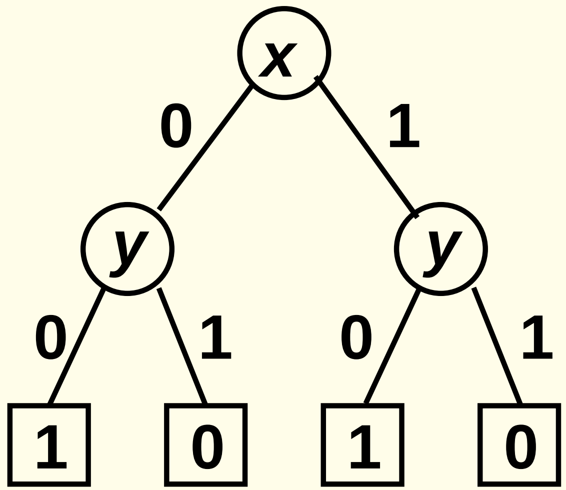
Binary Decision Diagrams

Formulae	$\leftrightarrow$	Boolean functions
F (n Prop.Var)	$\mapsto$	$f_F : \{0,1\}^n \rightarrow \{0,1\}$

Binary decision trees:



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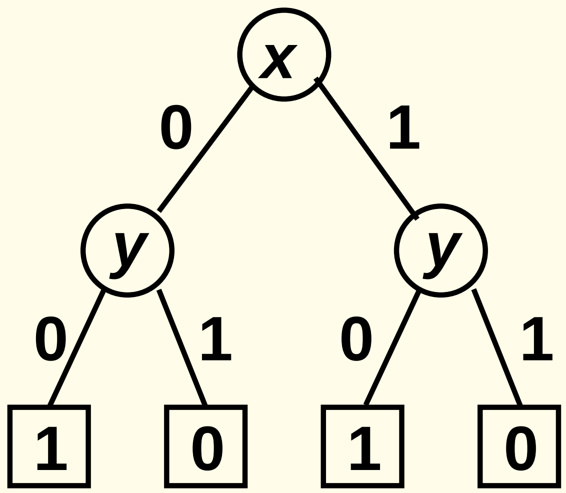

- exactly as inefficient as truth tables ($2^{n+1} - 1$ nodes if n prop.vars.)

- optimization possible: remove redundancies

Binary Decision Diagrams

Optimization: remove redundancies

1. remove duplicate leaves

2. remove unnecessary tests

3. remove duplicate nodes

Binary Decision Diagrams

Binary decision diagram (BDD): finite directed acyclic graph with:

- a unique initial node
- terminal nodes marked with 0 or 1
- non-terminal nodes marked with propositional variables
- in each non-terminal node: two vertices (marked 0/1)

Reduced BDD: Optimizations 1-3 cannot be applied.

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Problem: Variables may occur several times on a path.

Solution: Ordered BDDs.

Ordered BDDs

$[P_1, \ldots, P_n]$ ordered list of variables (without repetitions)

Let B be a BDD with variables $\{P_1, \ldots, P_n\}$

B has the order $[P_1, \ldots, P_n]$
if for every path $v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_m$ in B,
if - $i < j$,

- v_i is marked with P_{k_i}

- v_j ist marked with P_{k_j}

then $k_i < k_j$.

A ordered BDD (Notation: OBDD) is a BDD which has an order, for a certain ordered list of variables.

Reduced OBDDs

Let $[P_1, \ldots, P_n]$ be an order on variables.

The reduced OBDD, which represents a given function f is unique.

Theorem:

Let B_1, B_2 be two reduced OBDDs with the same variable ordering.

If B_1 and B_2 represent the same function, then B_1 and B_2 are equal.

OBDDs have a canonical form, namely the reduced OBDD.

The role of the ordering on variables

Example $(P_1 \vee P_2) \wedge (P_3 \vee P_4) \wedge \cdots \wedge (P_{2n-1} \vee P_{2n})$

$[P_1, P_2, \ldots, P_{2n-1}, P_{2n}]$: OBDD with $2n + 2$ nodes

$[P_1, P_3, \ldots, P_{2n-1}, P_2, \ldots, P_{2n}]$: OBDD with 2^{n+1} nodes

Advantages of canonical representations

- Absence of redundant variables

  If the value of f does not depend on the i-argument (P_i)
  then no reduced OBDD contains the variable P_i

- Equivalence test

  $F_i \mapsto f_i \mapsto B_i$ (OBDDs with compatible variable ordering), $i = 1, 2$

  Reduce B_i, $i = 1, 2$. $F_1 \equiv F_2$ iff. B_1 and B_2 identical.

Advantages of canonical representations

- Validity test

  $F \mapsto f \mapsto B$ (OBDD)

  F valid iff its reduced OBDD is $B_1 := \boxed{1}$

- Entailment test

  $F \models G$ iff the reduced OBDD for $F \wedge \neg G$ is $B_0 := \boxed{0}$

- Satisfiability test

  F satisfiable iff its reduced OBDD is not B_0.

Operations with OBDDs

- Reduce

  Apply reduction steps 1–3

- Apply

  Boolean operations

- Restrict

  Compute OBDD for $F[0/P_i]$ and $F[1/P_i]$

- Exists

  Compute OBDD for $\exists P_i F(P_1, \ldots, P_n)$

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Reduce

remove redundancies

1. remove duplicate leaves

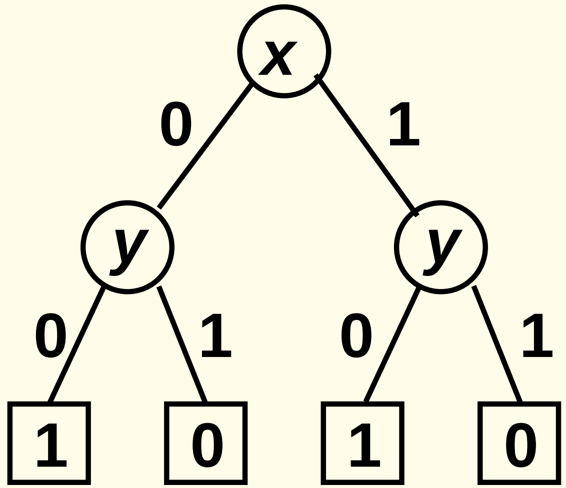
2. remove unnecessary tests

3. remove duplicate nodes

Reduce

1. remove duplicate leaves

Only one copy of 0 and 1 necessary:



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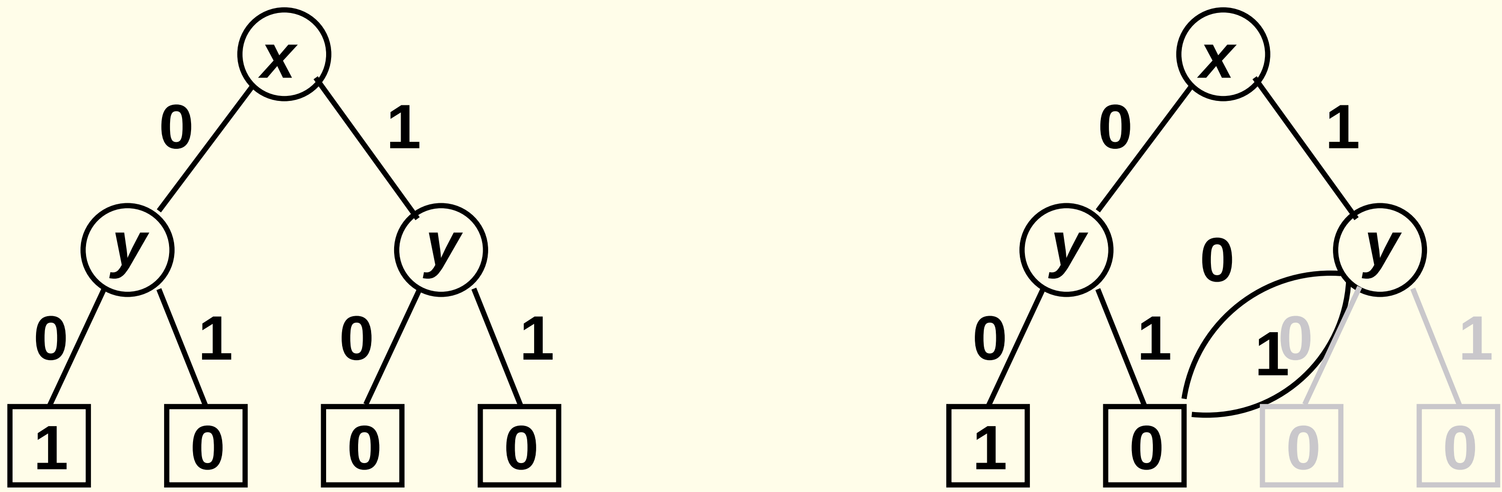
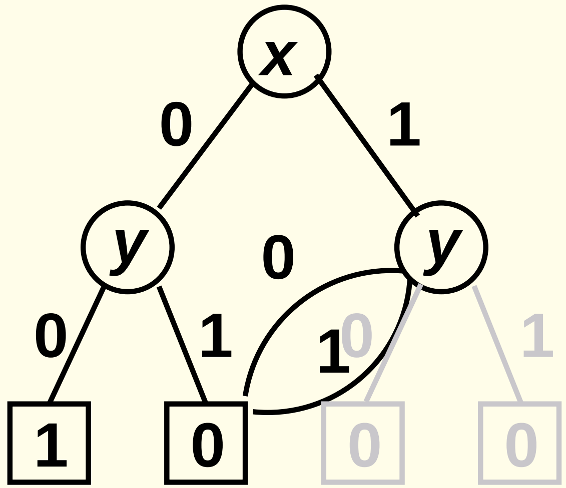


Reduce

2. remove unnecessary tests



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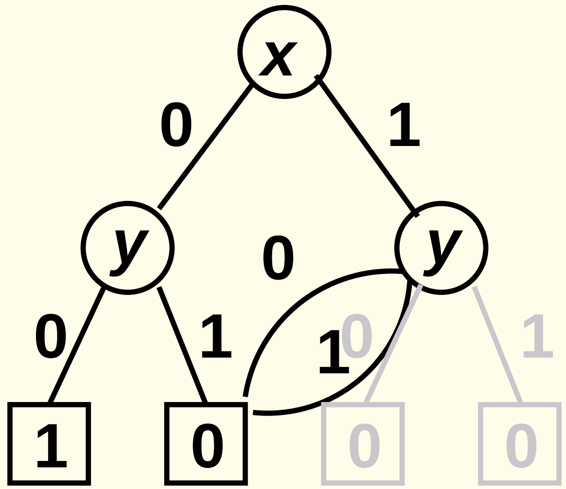
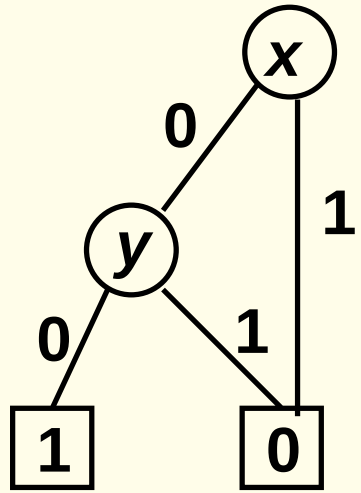
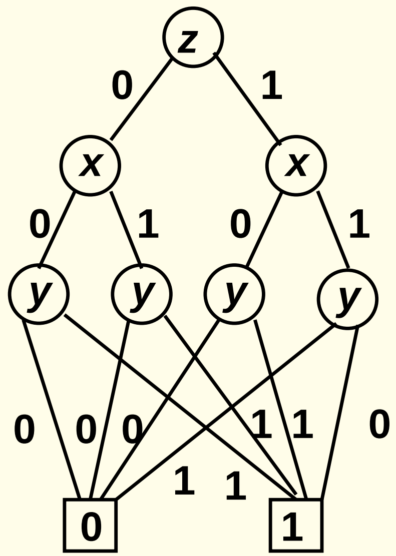




Reduce

3. remove duplicate non-terminal nodes:



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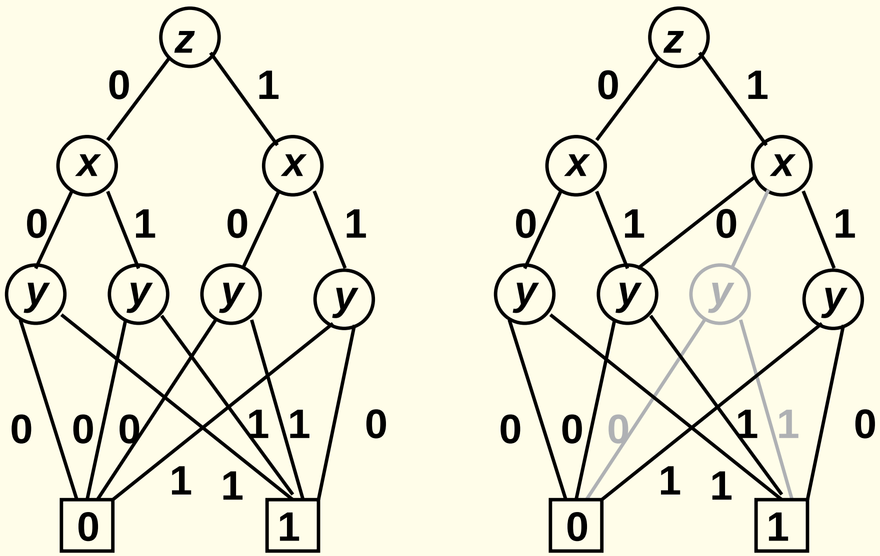

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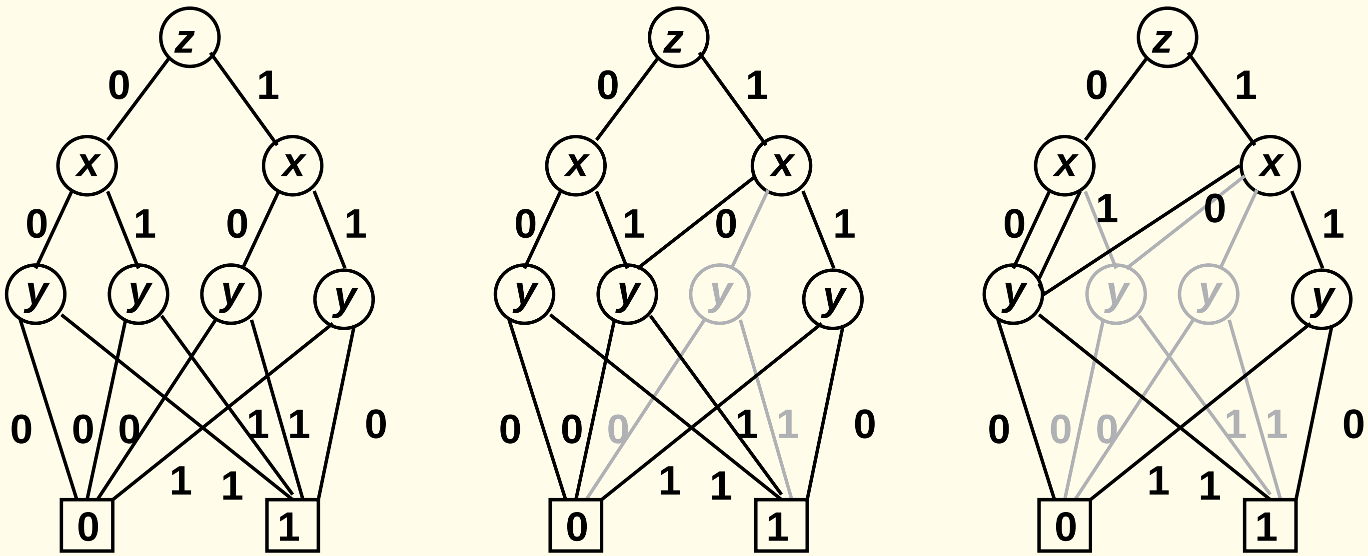

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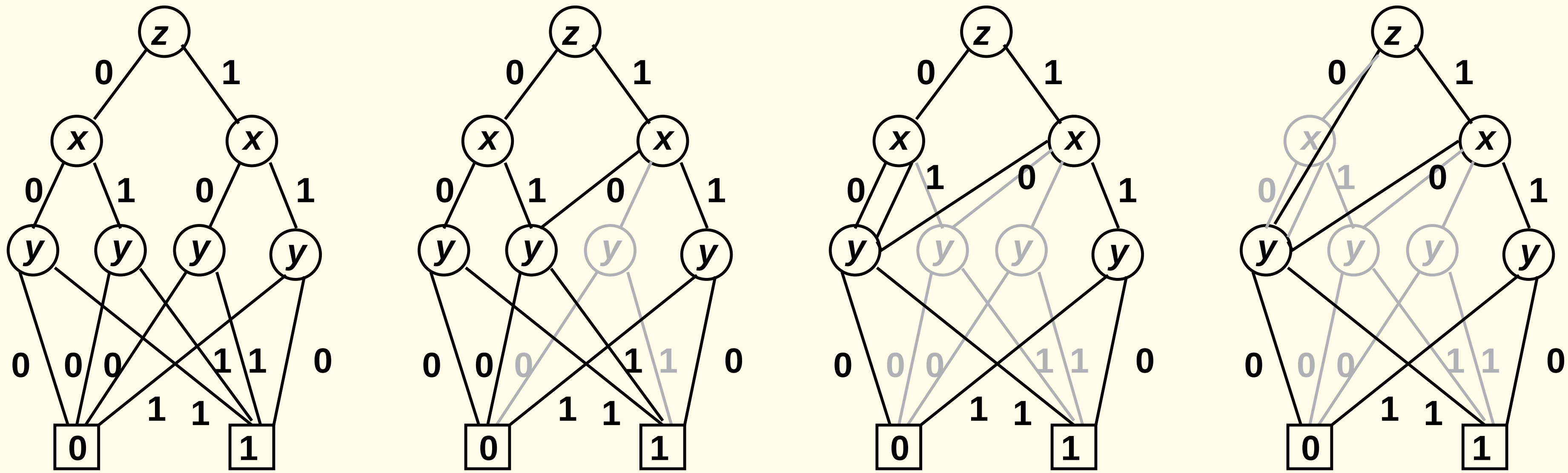

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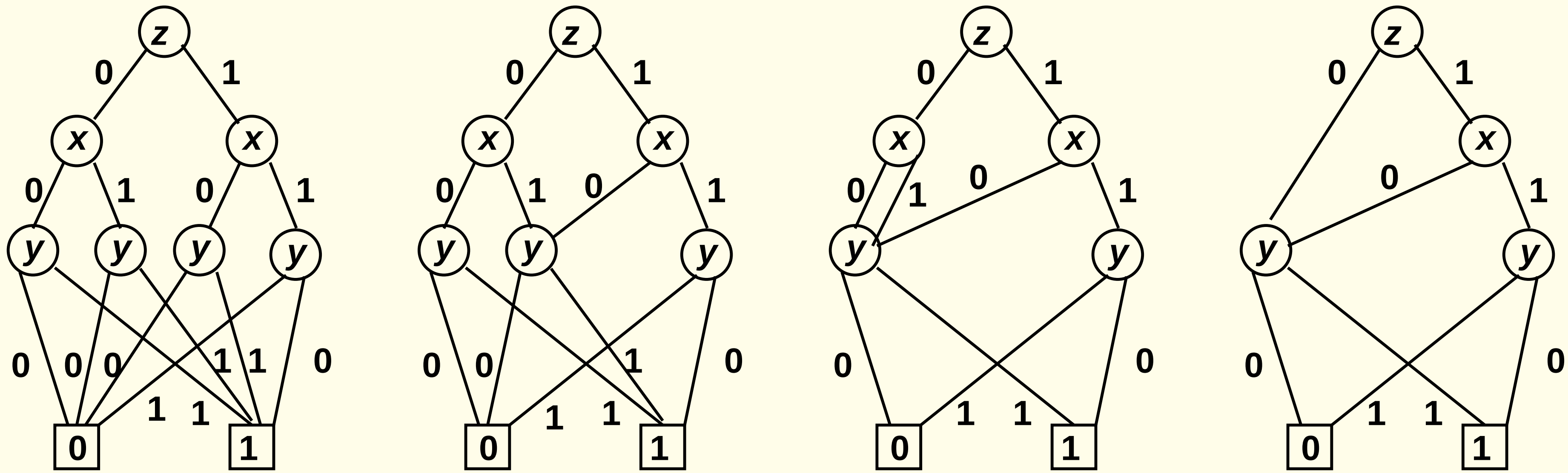

Reduce

The algorithm reduce traverses an OBDD B layer by layer in a bottom-up fashion, beginning with the terminal nodes.

In traversing B, it assigns an integer label $id(n)$ to each node n of B, in such a way that the subOBDDs with root nodes n and m denote the same boolean function iff, $id(n) = id(m)$.

Reduce

Terminal nodes:

Since reduce starts with the layer of terminal nodes, it assigns the first label (say 0) to the first 0-node it encounters. All other terminal 0-nodes denote the same function as the first 0-node and therefore get the same label (compare with reduction 1).

Similarly, the 1-nodes all get the next label, say 1.

Reduce

Non-terminal nodes

Now let us inductively assume that reduce has already assigned integer labels to all nodes of a layer $> i$ (i.e. all terminal nodes and P_j-nodes with $j > i$).

We describe how nodes of layer i (i.e. P_i-nodes) are being handled.

$n \mapsto lo(n)$ node reached on branch labelled with 0
$\quad hi(n)$ node reached on branch labelled with 1

Given an P_i-node n, there are three ways in which it may get its label:

- If $id(lo(n)) = id(hi(n))$, we set $id(n)$ to be that label (reduction 2)
- If there is another node m s.t. n and m have same variable P_i, and $id(lo(n)) = id(lo(m))$ and $id(hi(n)) = id(hi(m))$, then we set $id(n) := id(m)$ (reduction 3)
- Otherwise, we set $id(n)$ to the next unused integer label.

Operations with OBDDs

- Reduce

  Apply reduction steps 1–3

- Apply

  Boolean operations

- Restrict

  Compute OBDD for $F[0/P_i]$ and $F[1/P_i]$

- Exists

  Compute OBDD for $\exists P_i F(P_1, \ldots, P_n)$

Reminder: BDDs

$f \mapsto B_f$ (BDD associated with f)

$g \mapsto B_g$ (BDD associated with g)

BDD for $f \wedge g$: replace all 1-leaves in B_f with B_g

BDD for $f \vee g$: replace all 0-leaves in B_f with B_g

BDD for $\neg f$: replace all 1-leaves in B_f with 0-leaves and all 0-leaves with 1 leaves.

Reminder: BDDs

$f \mapsto B_f$ (BDD associated with f)

$g \mapsto B_g$ (BDD associated with g)

BDD for $f \wedge g$: replace all 1-leaves in B_f with B_g

BDD for $f \vee g$: replace all 0-leaves in B_f with B_g

BDD for $\neg f$: replace all 1-leaves in B_f with 0-leaves and all 0-leaves with 1 leaves.

If applied to OBDDs, the resulting BDD is not ordered!

Apply

Idea: Use the Shannon expansion for F.

$$F \equiv (\neg P \wedge F[0/P]) \vee (P \wedge F[1/P])$$

The function apply is based on the Shannon expansion for $F \mathrm{op} G$:

$$F \mathrm{op} G = (\neg P_i \wedge (F[0/P_i] \mathrm{op} G[0/P_i])) \vee (P_i \wedge (F[1/P_i] \mathrm{op} G[1/P_i])).$$

Apply

This is used as a control structure of apply which proceeds from the roots of B_F and B_G downwards to construct nodes of the OBDD $B_{F \mathrm{op} G}$.

Let r_f be the root node of B_F and r_g the root node of B_G.

1. If both r_f, r_g are terminal nodes with labels l_f and l_g, respectively (0 or 1), we compute the value $l_f \mathrm{op} l_g$ and let the resulting OBDD be B_0 if the value is 0 and B_1 otherwise.

Apply

This is used as a control structure of apply which proceeds from the roots of B_F and B_G downwards to construct nodes of the OBDD $B_{F \text{op} G}$.

Let r_f be the root node of B_F and r_g the root node of B_G.

In the remaining cases, at least one of the root nodes is a non-terminal.

2. Suppose that both root nodes are P_i-nodes.

   Then we create an P_i-node n with

   - the edge labelled with 0 to apply(op, lo(r_f), $lo(r_g)$)
   - the edge labelled with 1 to apply(op, hi(r_f), $hi(r_g)$)

Apply

This is used as a control structure of apply which proceeds from the roots of B_F and B_G downwards to construct nodes of the OBDD $B_{F \text{op} G}$.

Let r_f be the root node of B_F and r_g the root node of B_G.

3. If r_f is a P_i-node, but r_g is a terminal node or a P_j-node with $j > i$, then we know that there is no P_i-node in B_G (because the two OBDDs have a compatible ordering of boolean variables).

   Thus, G is independent of P_i ($G \equiv G[0/P_i] \equiv G[1/P_i]$).

   Therefore, we create a P_i-node n with: - the 0-edge to apply(op, $lo(r_f)$, r_g) and
   - the 1-edge to apply(op, $hi(r_f)$, r_g).

4. The case in which r_g is a non-terminal, but r_f is a terminal or a P_j-node with $j > i$, is handled symmetrically to case 3.

Apply

The result of this procedure might not be reduced; therefore apply finishes by calling the function reduce on the OBDD it constructed.

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  Compute OBDD for $\exists P_i F(P_1, \ldots, P_n)$

Restrict

Given an OBDD B_F representing a boolean formula F, we need an algorithm restrict such that:

– restrict$(0, P, B_F)$ computes the reduced OBDD for $F[0/P]$ using the same variable ordering as B_F.

The algorithm works as follows.

For each node n labelled with P, incoming edges are redirected to $lo(n)$ and n is removed.

Then we call reduce on the resulting OBDD.

The call restrict$(1, P, B_F)$ proceeds similarly, only we now redirect incoming edges to $hi(n)$.

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- Exists

  Compute OBDD for $\exists P_i F(P_1, \ldots, P_n)$

Exists

A boolean function can be thought of as putting a constraint on the values of its argument variables.

It is useful to be able to express the relaxation of the constraint on a subset of the variables concerned.

To allow this, we write $\exists P.F$ for the boolean function F with the constraint on P relaxed.

Formally, $\exists P.F$ is defined as $F[0/P] \vee F[1/P]$

that is, $\exists P.F$ is true if F could be made true by putting P to 0 or to 1.

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that is, $\exists P.F$ is true if F could be made true by putting P to 0 or to 1.

Therefore the exists algorithm can be implemented in terms of the algorithms apply and restrict as:

$$\text{exists}(P, F) := \text{apply}(\vee, \text{restrict}(0, P, B_F), \text{restrict}(1, P, B_F))$$