

Formal Specification and Verification

First order logic

Algorithmic Problems, Decidability, Undecidability

(Additional slides)

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Algorithmic Problems

Validity(F): $\models F$?

Satisfiability(F): F satisfiable?

Entailment(F, G): does F entail G ?

Model(\mathcal{A}, F): $\mathcal{A} \models F$?

Solve(\mathcal{A}, F): find an assignment β such that $\mathcal{A}, \beta \models F$

Solve(F): find a substitution σ such that $\models F\sigma$

Abduce(F): find G with “certain properties” such that G entails F

Decidability/Undecidability



In 1931, Gödel published his incompleteness theorems in “Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme” (in English “On Formally Undecidable Propositions of Principia Mathematica and Related Systems”).

He proved for any computable axiomatic system that is powerful enough to describe the arithmetic of the natural numbers (e.g. the Peano axioms or Zermelo-Fraenkel set theory with the axiom of choice), that:

- If the system is consistent, it cannot be complete.
- The consistency of the axioms cannot be proven within the system.

Decidability/Undecidability

These theorems ended a half-century of attempts, beginning with the work of Frege and culminating in Principia Mathematica and Hilbert's formalism, to find a set of axioms sufficient for all mathematics.

The incompleteness theorems also imply that not all mathematical questions are computable.

Consequences of Gödel's Famous Theorems

1. For most signatures Σ , validity is undecidable for Σ -formulas.
(One can easily encode Turing machines in most signatures.)
2. For each signature Σ , the set of valid Σ -formulas is recursively enumerable.
(We will prove this by giving complete deduction systems.)
3. For $\Sigma = \Sigma_{PA}$ and $\mathbb{N}_* = (\mathbb{N}, 0, s, +, *)$, the theory $Th(\mathbb{N}_*)$ is not recursively enumerable.

These undecidability results motivate the study of subclasses of formulas (**fragments**) of first-order logic

Q: Can you think of any fragments of first-order logic for which validity is decidable?

Some Decidable Fragments/Problems

Validity/Satisfiability/Entailment: Some decidable fragments:

- Variable-free formulas without equality: satisfiability is NP-complete. (why?)
- Variable-free Horn clauses (clauses with at most one positive atom): entailment is decidable in linear time.
- **Monadic class:** no function symbols, all predicates unary; validity is NEXPTIME-complete.
- Q: Other decidable fragments of FOL (with variables)? Which methods for proving decidability?

Decidable problems.

Finite model checking is decidable in time polynomial in the size of the structure and the formula.