Exercise 8.3 : Prove the following equivalences of CTL formulas:

(1) $\neg A\Diamond F \equiv E\Box\neg F$

By definition, $A\Box G = \neg E\Diamond\neg G$
$E\Box G = \neg A\Diamond\neg G$. for all formulae G .

$E\Box\neg F \overset{def}{:=} \neg A\Diamond\neg\neg F \equiv \neg A\Diamond F$.

(2) $\neg E\Diamond F \equiv A\Box\neg F$

$A\Box\neg F \overset{def}{:=} \neg E\Diamond\neg\neg F \equiv \neg E\Diamond F$.

(3) $\neg A\bigcirc F \equiv E\bigcirc\neg F$

To show: for all CTL structures $T=(S,\rightarrow,L)$ and all $s\in S$ we have

$(T,s)\models\neg A\bigcirc F$ iff $(T,s)\models E\bigcirc\neg F$.

Proof:
Let T be a CTL structure and s be a state.

$(T,s)\models\neg A\bigcirc F$ iff $(T,s)\not\models A\bigcirc F$
iff not (for all $t\in S$ with $s\rightarrow t$ we have $(T,t)\models F$)
iff there exists $t\in S$ with $s\rightarrow t$ and $(T,t)\not\models F$
iff there exists $t\in S$ with $s\rightarrow t$ and $(T,t)\models\neg F$
iff $(T,s)\models E\bigcirc\neg F$.

(4) $E(F\mathcal{U}G) \equiv G \vee (F \wedge EOE(F\mathcal{U}G))$

To show: for all CTL structures T & all states s of T

$(T,s) \models E(F\mathcal{U}G)$ iff $(T,s) \models G \vee (F \wedge EOE(F\mathcal{U}G))$

Proof Let T be a CTL structure and $s \in S$

$(T,s) \models E(F\mathcal{U}G)$ iff there exists a computation π of T
$\pi = s_0 \to s_1 \to s_2 \to \cdots$, $s_0 = s$
s.t. $\exists m \geq 0$ st. $(T,s_m) \models G$ and
$(T,s_k) \models F$ for all $k \in \{0..m-1\}$

iff $\exists$ computation $\pi = s_0 \to s_1 \to \cdots \to s_m$, $s_0 = s$
s.t. (i) $\exists m = 0$ st. $(T,s_m) \models G$ and $(T,s_k) \models F$ for $k < m$
or
$\exists$ computat. $\pi = s_0 \to s_1 \to \cdots \to s_m$, $s_0 = s$
st (ii) $\exists m \geq 1$ st $(T,s_m) \models G$ and $(T,s_k) \models F$ for $k < m$.

iff $\exists$ computation $\pi = s_0 \to s_1 \to \cdots$, $s_0 = s$
s.t (i') $(T,s_0) \models G$
$\exists$ computation $\pi = s_0 \to s_1 \to \cdots$ $s_0 = s$
st (ii') $(T,s_0) \models F$ and $(T,s_0) \models EOE(F\mathcal{U}G)$

it is easy to see that (i) and (i') are equivalent.
To show that (ii) and (ii') are equivalent we
note that:
Assume (ii) holds.
(1) if $m \geq 1$ & $(T,s_k) \models F$ for all $k < m$ then $(T,s_0) \models F$.
(2) (ii) implies that there exists a state
s_1 st. $(s \to s_1)$ and,
there exists a computation $\pi' = s_1 \to s_2 \to \cdots \to s_m \to \cdots$
such that $\exists m \geq 1$: $(T,s_m) \models G$ & $(T,s_k) \models F$
for all $k \in \{1..m-1\}$

But then there exists s_1 with $s \to s_1$ and
$(T,s_1) \models E(F\mathcal{U}G)$

So $(T,s_0) \models EOE(F\mathcal{U}G)$

Assume (ii') holds.

Then $\begin{cases} (T, s_0) \models F \text{ \&} \\ \text{there exists } s_1 \quad s \to s_1 \text{ and} \end{cases}$

there exists a path $s_1 \to s_2 \dashrightarrow s_n$

s.t. $(T, s_1) \models F \mathcal{U} G$ (i.e. $\exists m \geq 1$ s.t. $(T, s_m) \models G$ & $(T, s_k) \models F$ for all $k \in \{1 \ldots m-1\}$

Then: there exists a path $\pi = s_0 \to s_1 \to \cdots \to s_n$

$(T, s_0) \models F$ and

$\exists m \geq 1$: $(T, s_m) \models G$ and $(T, s_k) \models F$ for all $k \in \{1 \ldots m-1\}$

But this clearly implies (ii).

(5) $E\Box F \equiv F \wedge E \circ E\Box F$.

Let T be a CTL structure and s a state of T

$(T,s) \models E\Box F \iff (T,s) \models \neg A\Diamond\neg F$

$\iff (T,s) \not\models A(\top \, \mathcal{U} \, \neg F)$

$\iff$ not(for all paths $\pi = s_0 \to s_1 \to \dots s_n \to \dots$ with $s_0 = s$ then exists $m \geq 0$ with $(T,s_m) \models \neg F$)

(*) $\iff$ then exists a path $\pi = s_0 \to s_1 \to \dots \to s_n \to \dots$ with $s_0 = s$ s.t. for all $m \geq 0$ $(T,s_m) \models F$.

$(T,s) \models F \wedge E \circ E\Box F \iff$ $(T,s) \models F$ and there exists s_1 with $(s \to s_1)$ and $(T,s_1) \models E\Box F$.

(**) $\iff$ $(T,s) \models F$ and there exists s_1 with $s \to s_1$ and there exists a path $\pi_1 = s_1 \to s_2 \to \dots$ such that for all $m \geq 1$: $(T,s_m) \models F$.

(*) and (**) are clearly equivalent:

If (*) holds then $(T,s) \models F$ and there exists a path $\pi_1 = \pi^1$ on which F remains true, so (**) holds.

If (**) holds we can construct a path π adding $s_0 = s$ at the beginning of path π_1 which exists because of (ii). Then F is true at all states on path π. so (*) holds.