Formal Specification and Verification

6.05.2014

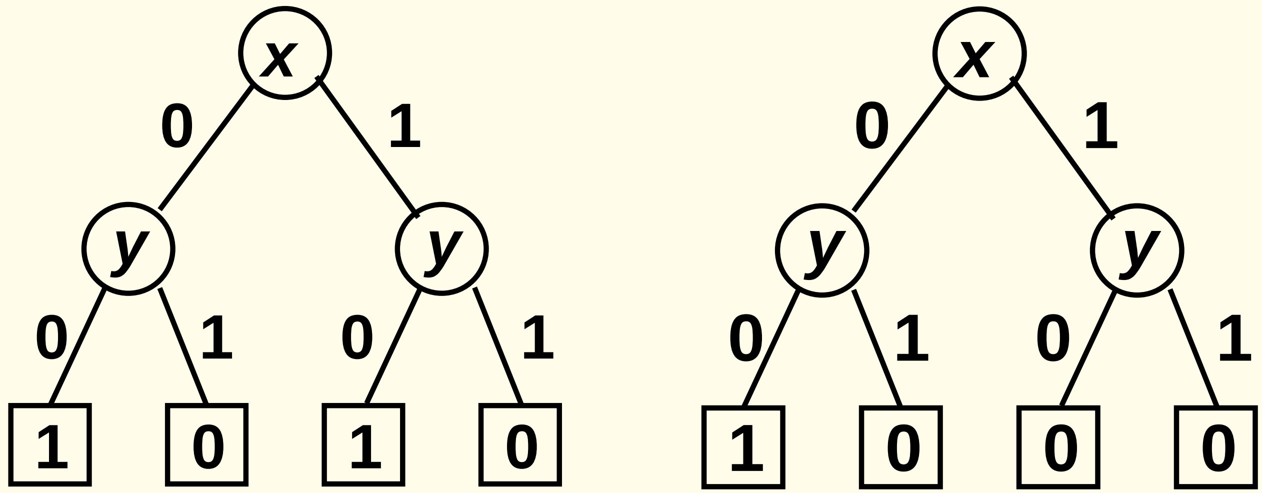
Viorica Sofronie-Stokkermans

e-mail: sofronie@uni-koblenz.de

Binary Decision Diagrams

Formulae	$\leftrightarrow$	Boolean functions
F (n Prop.Var)	$\mapsto$	$f_F : \{0,1\}^n \to \{0,1\}$

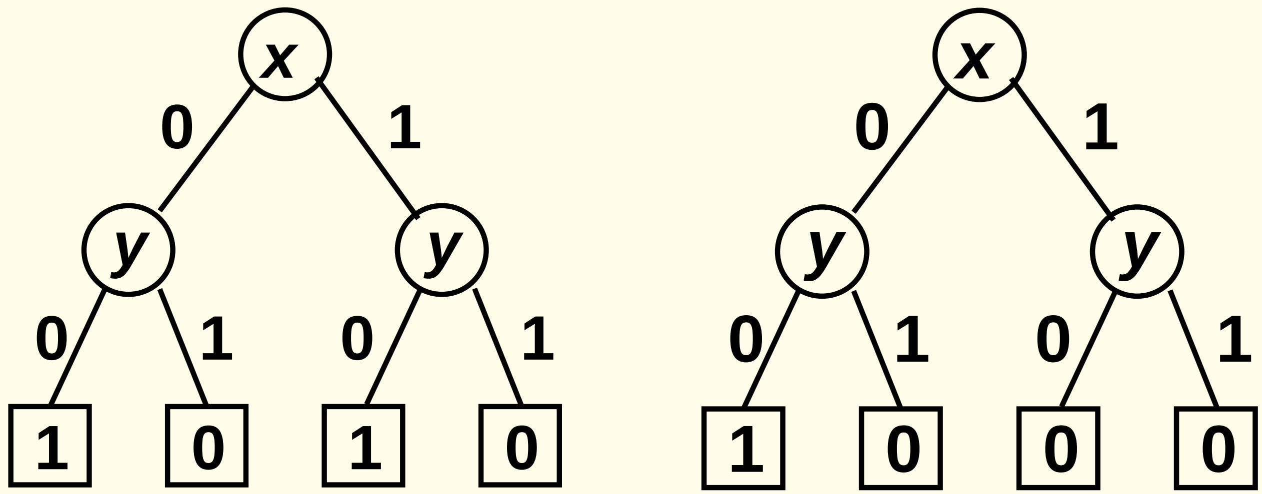
Binary decision trees:



Binary Decision Diagrams

Formulae	$\leftrightarrow$	Boolean functions
F (n Prop.Var)	$\mapsto$	$f_F : \{0,1\}^n \rightarrow \{0,1\}$

Binary decision trees:



- exactly as inefficient as truth tables ($2^{n+1} - 1$ nodes if n prop.vars.)

- optimization possible: remove redundancies

Binary Decision Diagrams

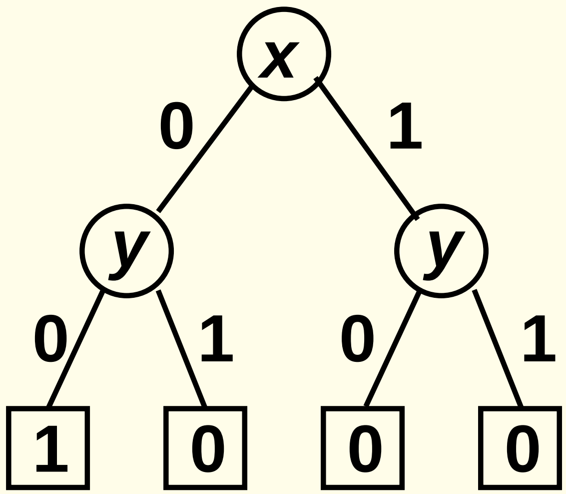
Optimization: remove redundancies

1. remove duplicate leaves
2. remove unnecessary tests
3. remove duplicate nodes

Binary Decision Diagrams

1. remove duplicate leaves

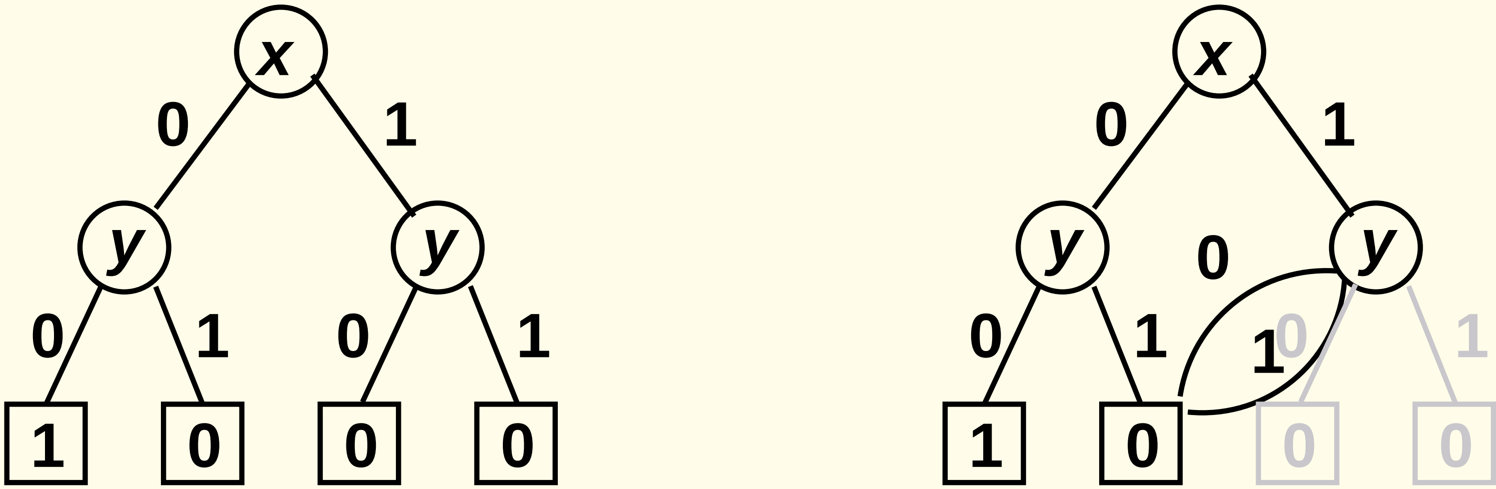
Only one copy of 0 and 1 necessary:



Binary Decision Diagrams

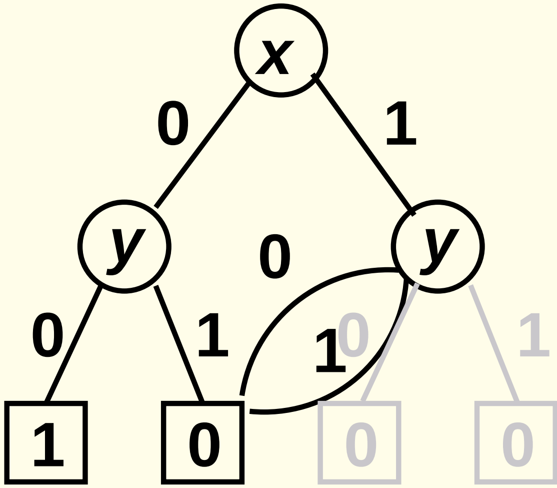
1. remove duplicate leaves

Only one copy of 0 and 1 necessary:



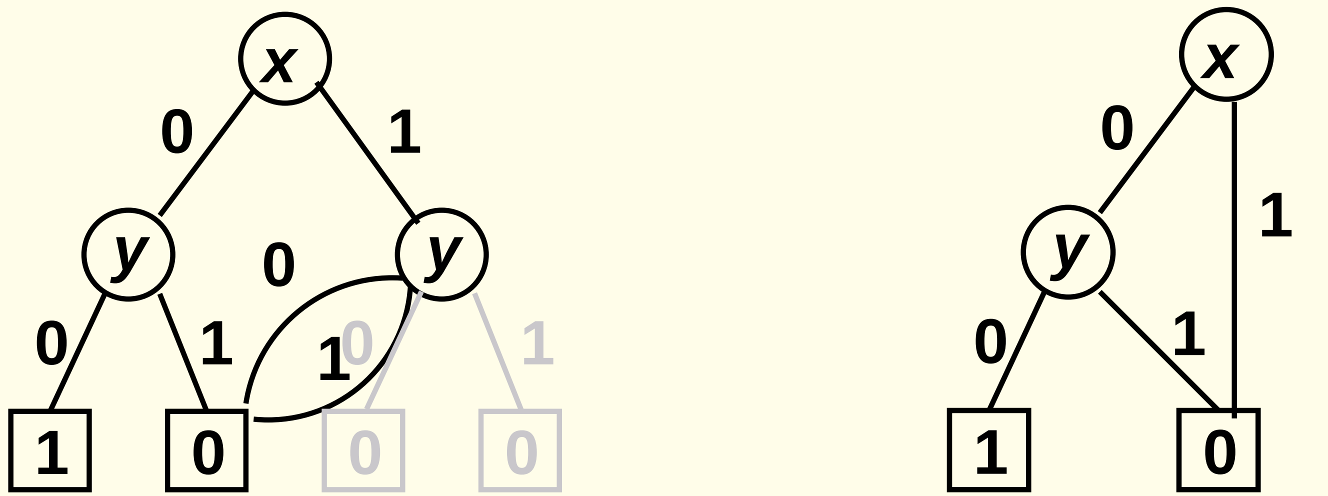
Binary Decision Diagrams

2. remove unnecessary tests



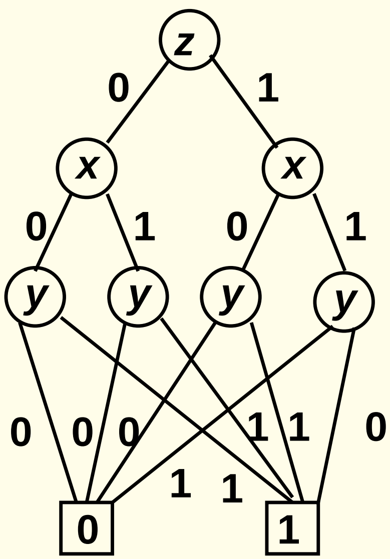
Binary Decision Diagrams

2. remove unnecessary tests



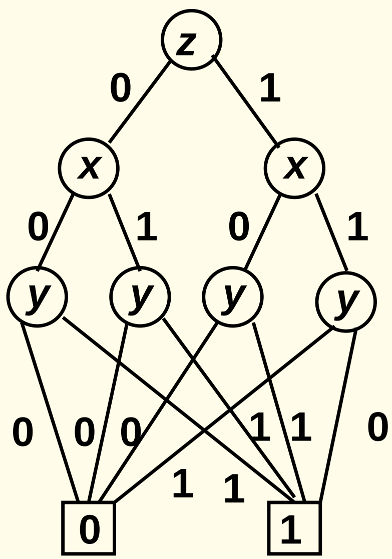
Binary Decision Diagrams

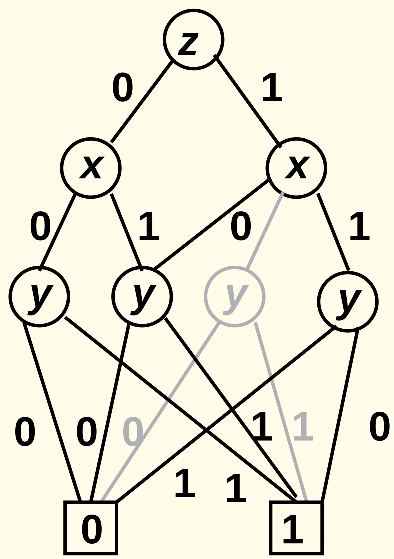
3. remove duplicate non-terminal nodes:



Binary Decision Diagrams

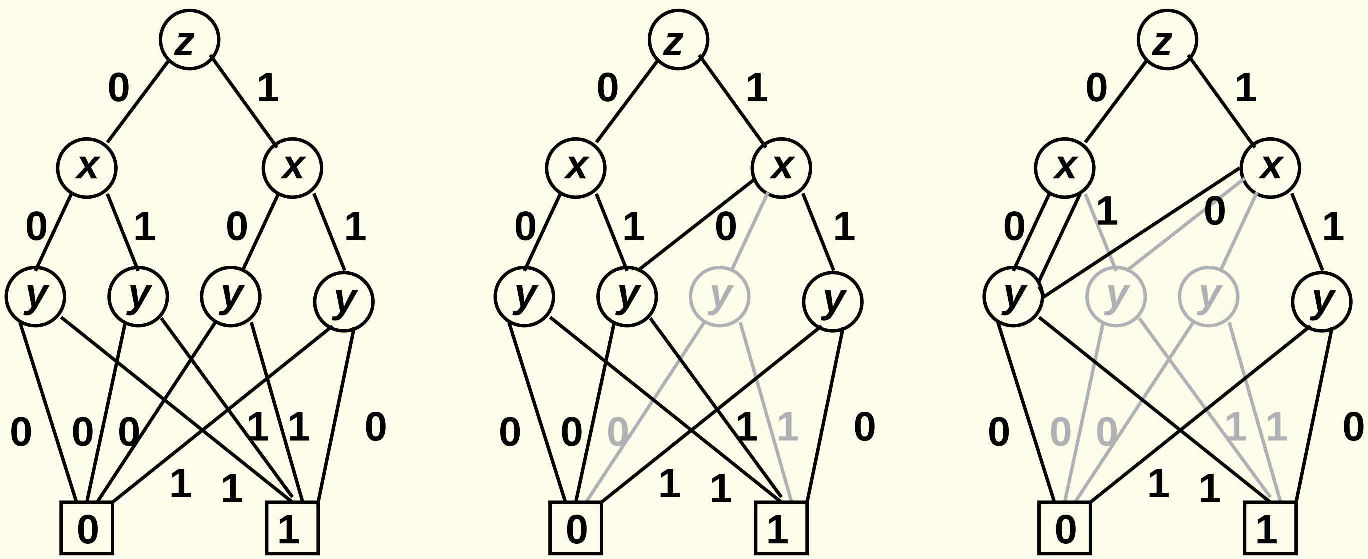
3. remove duplicate non-terminal nodes:





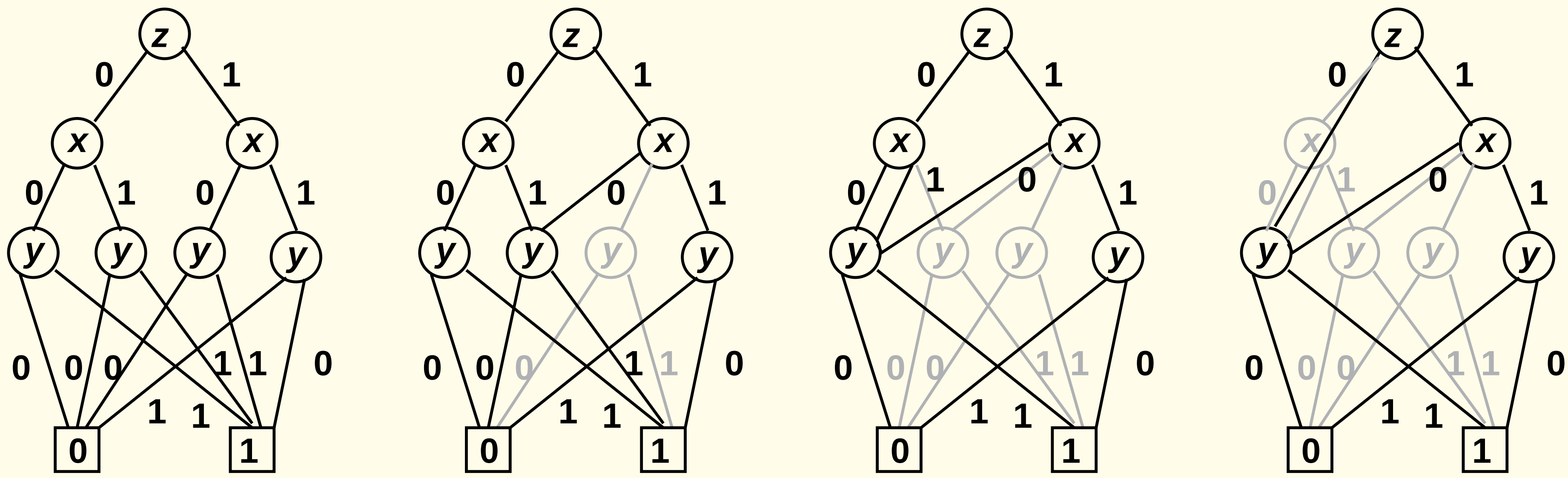
Binary Decision Diagrams

3. remove duplicate non-terminal nodes:



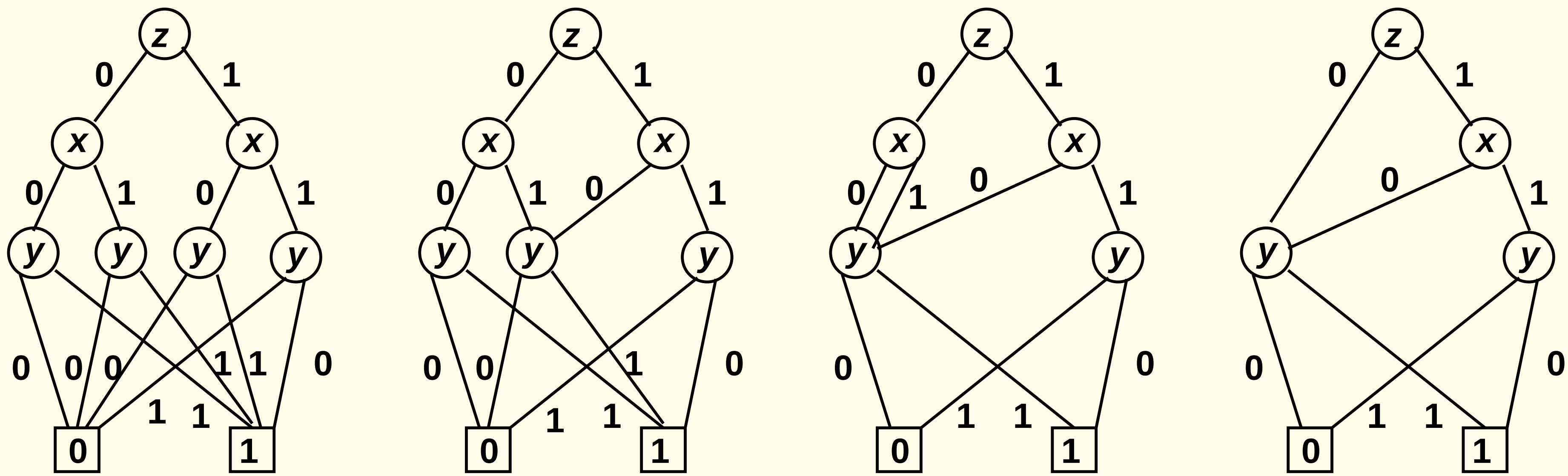
Binary Decision Diagrams

3. remove duplicate non-terminal nodes:



Binary Decision Diagrams

3. remove duplicate non-terminal nodes:



Operations with BDDs

$f \mapsto B_f$ (BDD associated with f)

$g \mapsto B_g$ (BDD associated with g)

BDD for $f \wedge g$: replace all 1-leaves in B_f with B_g

BDD for $f \vee g$: replace all 0-leaves in B_f with B_g

BDD for $\neg f$: replace all 1-leaves in B_f with 0-leaves and all 0-leaves with 1 leaves.

Binary Decision Diagrams

Binary decision diagram (BDD): finite directed acyclic graph with:

- a unique initial node
- terminal nodes marked with 0 or 1
- non-terminal nodes marked with propositional variables
- in each non-terminal node: two vertices (marked 0/1)

Reduced BDD: Optimizations 1-3 cannot be applied.

Binary Decision Diagrams

Binary decision diagram (BDD): finite directed acyclic graph with:

- a unique initial node
- terminal nodes marked with 0 or 1
- non-terminal nodes marked with propositional variables
- in each non-terminal node: two vertices (marked 0/1)

Reduced BDD: Optimizations 1-3 cannot be applied.

Problem: Variables may occur several times on a path.

Solution: Ordered BDDs.

Ordered BDDs

$[P_1, \ldots, P_n]$ ordered list of variables (without repetitions)

Let B be a BDD with variables $\{P_1, \ldots, P_n\}$

B has the order $[P_1, \ldots, P_n]$
if for every path $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_m$ in B,
if - $i < j$,

- v_i is marked with P_{k_i}

- v_j ist marked with P_{k_j}

then $k_i < k_j$.

A ordered BDD (Notation: OBDD) is a BDD which has an order, for a certain ordered list of variables.

Reduced OBDDs

Let $[P_1, \ldots, P_n]$ be an order on variables.

The reduced OBDD, which represents a given function f is unique.

Theorem:

Let B_1, B_2 be two reduced OBDDs with the same variable ordering.

If B_1 and B_2 represent the same function, then B_1 and B_2 are equal.

OBDDs have a canonical form, namely the reduced OBDD.

Advantages of canonical representations

- Absence of redundant variables

  If the value of f does not depend on the i-argument (P_i)
  then no reduced OBDD contains the variable P_i

- Equivalence test

  $F_i \mapsto f_i \mapsto B_i$ (OBDDs with compatible variable ordering), $i = 1, 2$

  Reduce B_i, $i = 1, 2$. $F_1 \equiv F_2$ iff. B_1 and B_2 identical.

Advantages of canonical representations

- Validity test

  $F \mapsto f \mapsto B$ (OBDD)

  F valid iff its reduced OBDD is $B_1 := \boxed{1}$

- Entailment test

  $F \models G$ iff the reduced OBDD for $F \wedge \neg G$ is $B_0 := \boxed{0}$

- Satisfiability test

  F satisfiable iff its reduced OBDD is not B_0.

Operations with OBDDs

- Reduce

  Apply reduction steps 1–3

- Apply

  Boolean operations

- Restrict

  Compute OBDD for $F[0/P_i]$ and $F[1/P_i]$

- Exists

  Compute OBDD for $\exists P_i F(P_1, \ldots, P_n)$

Operations with OBDDs

- Reduce

  Apply reduction steps 1–3

- Apply

  Boolean operations

- Restrict

  Compute OBDD for $F[0/P_i]$ and $F[1/P_i]$

- Exists

  Compute OBDD for $\exists P_i F(P_1, \ldots, P_n)$

Reduce

remove redundancies

1. remove duplicate leaves

2. remove unnecessary tests

3. remove duplicate nodes

Reduce

The algorithm reduce traverses an OBDD B layer by layer in a bottom-up fashion, beginning with the terminal nodes.

In traversing B, it assigns an integer label $id(n)$ to each node n of B, in such a way that the subOBDDs with root nodes n and m denote the same boolean function iff, $id(n) = id(m)$.

Reduce

Terminal nodes:

Since reduce starts with the layer of terminal nodes, it assigns the first label (say $\#0$) to the first 0-node it encounters. All other terminal 0-nodes denote the same function as the first 0-node and therefore get the same label (compare with reduction 1).

Similarly, the 1-nodes all get the next label, say $\#1$.

Reduce

Non-terminal nodes

Now let us inductively assume that reduce has already assigned integer labels to all nodes of a layer $> i$ (i.e. all terminal nodes and P_j-nodes with $j > i$).

We describe how nodes of layer i (i.e. P_i-nodes) are being handled.

$n \mapsto lo(n)$ node reached on branch labelled with 0
$\quad hi(n)$ node reached on branch labelled with 1

Given an P_i-node n, there are three ways in which it may get its label:

- If $id(lo(n)) = id(hi(n))$, we set $id(n)$ to be that label (reduction 2)
- If there is another node m s.t. n and m have same variable P_i, and $id(lo(n)) = id(lo(m))$ and $id(hi(n)) = id(hi(m))$, then we set $id(n) := id(m)$ (reduction 3)
- Otherwise, we set $id(n)$ to the next unused integer label.

Operations with OBDDs

- Reduce

  Apply reduction steps 1–3

- Apply

  Boolean operations

- Restrict

  Compute OBDD for $F[0/P_i]$ and $F[1/P_i]$

- Exists

  Compute OBDD for $\exists P_i F(P_1, \ldots, P_n)$

Reminder: BDDs

$f \mapsto B_f$ (BDD associated with f)

$g \mapsto B_g$ (BDD associated with g)

BDD for $f \wedge g$: replace all 1-leaves in B_f with B_g

BDD for $f \vee g$: replace all 0-leaves in B_f with B_g

BDD for $\neg f$: replace all 1-leaves in B_f with 0-leaves and all 0-leaves with 1 leaves.

Reminder: BDDs

$f \mapsto B_f$ (BDD associated with f)

$g \mapsto B_g$ (BDD associated with g)

BDD for $f \wedge g$: replace all 1-leaves in B_f with B_g

BDD for $f \vee g$: replace all 0-leaves in B_f with B_g

BDD for $\neg f$: replace all 1-leaves in B_f with 0-leaves and all 0-leaves with 1 leaves.

If applied to OBDDs, the resulting BDD is not ordered!

Apply

Idea: Use the Shannon expansion for F.

$$F \equiv (\neg P \wedge F[0/P]) \vee (P \wedge F[1/P])$$

The function apply is based on the Shannon expansion for $F \mathrm{op} G$:

$$F \mathrm{op} G = (\neg P_i \wedge (F[0/P_i] \mathrm{op} G[0/P_i])) \vee (P_i \wedge (F[1/P_i] \mathrm{op} G[1/P_i])).$$

Apply

This is used as a control structure of apply which proceeds from the roots of B_F and B_G downwards to construct nodes of the OBDD $B_{F\mathrm{op}G}$.

Let r_f be the root node of B_F and r_g the root node of B_G.

1. If both r_f, r_g are terminal nodes with labels l_f and l_g, respectively (0 or 1), we compute the value $l_f \mathrm{op} l_g$ and let the resulting OBDD be B_0 if the value is 0 and B_1 otherwise.

Apply

This is used as a control structure of apply which proceeds from the roots of B_F and B_G downwards to construct nodes of the OBDD $B_{F \mathrm{op} G}$.

Let r_f be the root node of B_F and r_g the root node of B_G.

In the remaining cases, at least one of the root nodes is a non-terminal.

2. Suppose that both root nodes are P_i-nodes.

   Then we create an P_i-node n with

   - the edge labelled with 0 to apply(op, $lo(r_f)$, $lo(r_g)$)
   - the edge labelled with 1 to apply(op, $hi(r_f)$, $hi(r_g)$)

Apply

This is used as a control structure of apply which proceeds from the roots of B_F and B_G downwards to construct nodes of the OBDD $B_{F\text{op}G}$.

Let r_f be the root node of B_F and r_g the root node of B_G.

3. If r_f is a P_i-node, but r_g is a terminal node or a P_j-node with $j > i$, then we know that there is no P_i-node in B_G (because the two OBDDs have a compatible ordering of boolean variables).

   Thus, G is independent of P_i ($G \equiv G[0/P_i] \equiv G[1/P_i]$).

   Therefore, we create a P_i-node n with: - the 0-edge to apply(op, $lo(r_f)$, r_g) and
   - the 1-edge to apply(op, $hi(r_f)$, r_g).

4. The case in which r_g is a non-terminal, but r_f is a terminal or a P_j-node with $j > i$, is handled symmetrically to case 3.

Apply

The result of this procedure might not be reduced; therefore apply finishes by calling the function reduce on the OBDD it constructed.

Restrict

Given an OBDD B_F representing a boolean formula F, we need an algorithm restrict such that:

– restrict$(0, P, B_F)$ computes the reduced OBDD for $F[0/P]$ using the same variable ordering as B_F.

The algorithm works as follows.

For each node n labelled with P, incoming edges are redirected to $lo(n)$ and n is removed.

Then we call reduce on the resulting OBDD.

The call restrict$(1, P, B_F)$ proceeds similarly, only we now redirect incoming edges to $hi(n)$.

Operations with OBDDs

- Reduce

  Apply reduction steps 1–3

- Apply

  Boolean operations

- Restrict

  Compute OBDD for $F[0/P_i]$ and $F[1/P_i]$

- Exists

  Compute OBDD for $\exists P_i F(P_1, \ldots, P_n)$

Exists

A boolean function can be thought of as putting a constraint on the values of its argument variables.

It is useful to be able to express the relaxation of the constraint on a subset of the variables concerned.

To allow this, we write $\exists P.F$ for the boolean function F with the constraint on P relaxed.

Formally, $\exists P.F$ is defined as $F[0/P] \vee F[1/P]$

that is, $\exists P.F$ is true if F could be made true by putting P to 0 or to 1.

Exists

Formally, $\exists P.F$ is defined as $F[0/P] \vee F[1/P]$

that is, $\exists P.F$ is true if F could be made true by putting P to 0 or to 1.

Therefore the exists algorithm can be implemented in terms of the algorithms apply and restrict as:

$$\text{exists}(P, F) := \text{apply}(\vee, \text{restrict}(0, P, B_F), \text{restrict}(1, P, B_F))$$

Limitations of Propositional Logic

- Fixed, finite number of objects
  Cannot express: let G be group with arbitrary number of elements
- No functions or relations with arguments
  Can express: finite function/relation table p_{ij}
  Cannot express: properties of function/relation on all arguments, e.g., $+$ is associative
- Static interpretation
  Programs change value of their variables, e.g., via assignment, call, etc.
  Propositional formulas look at one single interpretation at a time

Beyond the Limitations of Propositional Logic

- First order logic

  (+ functions)

- Temporal logic

  (+ computations)

- Dynamic logic

  (+ computations + functions)

Part 2: First-Order Logic

→ First-order logic

- formalizes fundamental mathematical concepts
- is expressive (Turing-complete)
- is not too expressive
  (e. g. not axiomatizable: natural numbers, uncountable sets)
- has a rich structure of decidable fragments
- has a rich model and proof theory

First-order logic is also called (first-order) predicate logic.

2.1 Syntax

Syntax:

- non-logical symbols (domain-specific)
  $\Rightarrow$ terms, atomic formulas
- logical symbols (domain-independent)
  $\Rightarrow$ Boolean combinations, quantifiers

Signature

A signature

$$\Sigma = (\Omega, \Pi),$$

fixes an alphabet of non-logical symbols, where

- Ω is a set of function symbols f with arity $n \geq 0$, written f/n,
- Π is a set of predicate symbols p with arity $m \geq 0$, written p/m.

If $n = 0$ then f is also called a constant (symbol).
If $m = 0$ then p is also called a propositional variable.
We use letters P, Q, R, S, to denote propositional variables.

Signature

Refined concept for practical applications:
many-sorted signatures (corresponds to simple type systems in programming languages).

Most results established for one-sorted signatures extend in a natural way to many-sorted signatures.

Many-sorted Signature

A many-sorted signature

$$\Sigma = (S, \Omega, \Pi),$$

fixes an alphabet of non-logical symbols, where

- S is a set of sorts,
- Ω is a set of function symbols f with arity $a(f) = s_1 \ldots s_n \to s$,
- Π is a set of predicate symbols p with arity $a(p) = s_1 \ldots s_m$

where $s_1, \ldots, s_n, s_m, s$ are sorts.

Variables

Predicate logic admits the formulation of abstract, schematic assertions. (Object) variables are the technical tool for schematization.

We assume that

$$X$$

is a given countably infinite set of symbols which we use for (the denotation of) variables.

Variables

Predicate logic admits the formulation of abstract, schematic assertions. (Object) variables are the technical tool for schematization.

We assume that

$$X$$

is a given countably infinite set of symbols which we use for (the denotation of) variables.

Many-sorted case:

We assume that for every sort $s \in S$, X_s is a given countably infinite set of symbols which we use for (the denotation of) variables of sort s.

Terms

Terms over Σ (resp., Σ-terms) are formed according to these syntactic rules:

$$
\begin{array}{rcllr}
t, u, v & ::= & x & , x \in X & \text{(variable)} \\
 & | & f(t_1, ..., t_n) & , f/n \in \Omega & \text{(functional term)}
\end{array}
$$

By $\mathrm{T}_\Sigma(X)$ we denote the set of Σ-terms (over X).
A term not containing any variable is called a ground term.
By T_Σ we denote the set of Σ-ground terms.

Terms

Terms over Σ (resp., Σ-terms) are formed according to these syntactic rules:

$$
\begin{array}{rclll}
t, u, v & ::= & x & , x \in X & \text{(variable)} \\
 & | & f(t_1, ..., t_n) & , f/n \in \Omega & \text{(functional term)}
\end{array}
$$

By $\mathrm{T}_\Sigma(X)$ we denote the set of Σ-terms (over X).
A term not containing any variable is called a ground term.
By T_Σ we denote the set of Σ-ground terms.

Many-sorted case:

a variable $x \in X_s$ is a term of sort s

if $a(f) = s_1 \dots s_n \to s$, and t_i are terms of sort s_i, $i = 1, \dots, n$ then $f(t_1, ..., t_n)$ is a term of sort s.

Terms

In other words, terms are formal expressions with well-balanced brackets which we may also view as marked, ordered trees.

The markings are function symbols or variables.

The nodes correspond to the subterms of the term.

A node v that is marked with a function symbol f of arity n has exactly n subtrees representing the n immediate subterms of v.

Atoms

Atoms (also called atomic formulas) over Σ are formed according to this syntax:

$$
\begin{array}{llll}
A, B & ::= & p(t_1, \ldots, t_m) & ,\ p/m \in \Pi \\
\Big[& \mid & (t \approx t') & \text{(equation)} \quad \Big]
\end{array}
$$

Whenever we admit equations as atomic formulas we are in the realm of first-order logic with equality. Admitting equality does not really increase the expressiveness of first-order logic, (cf. exercises). But deductive systems where equality is treated specifically can be much more efficient.

Atoms

Atoms (also called atomic formulas) over Σ are formed according to this syntax:

$$\begin{array}{llll} A, B & ::= & p(t_1, ..., t_m) & ,\ p/m \in \Pi \\ \Big[& \mid & (t \approx t') & \text{(equation)} \quad \Big] \end{array}$$

Whenever we admit equations as atomic formulas we are in the realm of first-order logic with equality. Admitting equality does not really increase the expressiveness of first-order logic, (cf. exercises). But deductive systems where equality is treated specifically can be much more efficient.

Many-sorted case:

If $a(p) = s_1 \ldots s_m$, we require that t_i is a term of sort s_i for $i = 1, \ldots, m$.

Literals

$$
\begin{array}{lcll}
L & ::= & A & \text{(positive literal)} \\
 & \mid & \neg A & \text{(negative literal)}
\end{array}
$$

Clauses

$$
\begin{array}{rcll}
C, D & ::= & \bot & \text{(empty clause)} \\
 & \mid & L_1 \vee \ldots \vee L_k, \quad k \geq 1 & \text{(non-empty clause)}
\end{array}
$$

General First-Order Formulas

$F_\Sigma(X)$ is the set of first-order formulas over Σ defined as follows:

F, G, H	$::=$	$\bot$	(falsum)
	$\mid$	$\top$	(verum)
	$\mid$	A	(atomic formula)
	$\mid$	$\neg F$	(negation)
	$\mid$	$(F \wedge G)$	(conjunction)
	$\mid$	$(F \vee G)$	(disjunction)
	$\mid$	$(F \rightarrow G)$	(implication)
	$\mid$	$(F \leftrightarrow G)$	(equivalence)
	$\mid$	$\forall x F$	(universal quantification)
	$\mid$	$\exists x F$	(existential quantification)

Notational Conventions

We omit brackets according to the following rules:

- $\neg \quad >_p \quad \wedge \quad >_p \quad \vee \quad >_p \quad \rightarrow \quad >_p \quad \leftrightarrow$
  (binding precedences)
- $\vee$ and $\wedge$ are associative and commutative
- $\rightarrow$ is right-associative

$Qx_1, \ldots, x_n\, F$ abbreviates $Qx_1 \ldots Qx_n\, F$.

Notational Conventions

We use infix-, prefix-, postfix-, or mixfix-notation with the usual operator precedences.

Examples:

$s + t * u$	for	$+(s, *(t, u))$
$s * u \leq t + v$	for	$\leq (*(s, u), +(t, v))$
$-s$	for	$-(s)$
0	for	$0()$

Example: Peano Arithmetic

Signature:

$$
\begin{aligned}
\Sigma_{PA} &= (\Omega_{PA},\ \Pi_{PA}) \\
\Omega_{PA} &= \{0/0,\ +/2,\ */2,\ s/1\} \\
\Pi_{PA} &= \{\leq /2,\ < /2\}
\end{aligned}
$$

$+$, $*$, $<$, $\leq$ infix; $* \ >_p \ + \ >_p \ < \ >_p \ \leq$

Examples of formulas over this signature are:

$$\forall x, y(x \leq y \leftrightarrow \exists z(x + z \approx y))$$
$$\exists x \forall y(x + y \approx y)$$
$$\forall x, y(x * s(y) \approx x * y + x)$$
$$\forall x, y(s(x) \approx s(y) \rightarrow x \approx y)$$
$$\forall x \exists y(x < y \wedge \neg \exists z(x < z \wedge z < y))$$

Remarks About the Example

We observe that the symbols $\leq$, $<$, 0, s are redundant as they can be defined in first-order logic with equality just with the help of $+$. The first formula defines $\leq$, while the second defines zero. The last formula, respectively, defines s.

Eliminating the existential quantifiers by Skolemization (cf. below) reintroduces the "redundant" symbols.

Consequently there is a *trade-off* between the complexity of the quantification structure and the complexity of the signature.

Example: Specifying LISP lists

Signature:

$\Sigma_{\text{Lists}} = (\Omega_{\text{Lists}}, \Pi_{\text{Lists}})$

$\Omega_{\text{Lists}} = \{\text{car}/1, \text{cdr}/1, \text{cons}/2\}$

$\Pi_{\text{Lists}} = \emptyset$

Examples of formulae:

$\forall x, y \;\; \text{car}(\text{cons}(x, y)) \approx x$

$\forall x, y \;\; \text{cdr}(\text{cons}(x, y)) \approx y$

$\forall x \quad \text{cons}(\text{car}(x), \text{cdr}(x)) \approx x$

Many-sorted signatures

Example:

Signature

$S = \{\text{array}, \text{index}, \text{element}\}$ set of sorts

$\Omega = \{\text{read}, \text{write}\}$

$$a(\text{read}) = \text{array} \times \text{index} \to \text{element}$$
$$a(\text{write}) = \text{array} \times \text{index} \times \text{element} \to \text{array}$$

$\Pi = \emptyset$

$X = \{X_s \mid s \in S\}$

Examples of formulae:

$\forall x : \text{array} \ \ \forall i : \text{index} \ \ \forall j : \text{index} \ \ (i \approx j \to \text{write}(x, i, \text{read}(x, j)) \approx x)$

$\forall x : \text{array} \ \forall y : \text{array} \ \ (x \approx y \leftrightarrow \forall i : \text{index} \ \ (\text{read}(x, i) \approx \text{read}(y, i)))$