Formal Specification and Verification

Temporal logic (3)

24.06.2014

Viorica Sofronie-Stokkermans

e-mail: sofronie@uni-koblenz.de

Temporal logic

Linear Time Logic

Syntax

Π set of propositional variables.

The set of LTL (linear time logic) formulae is the smallest set such that:

- $\bot, \top$ and each propositional variable $P \in \Pi$ are formulae;
- if F, G are formulae, then so are $F \wedge G, F \vee G, \neg F$;
- if F, G are formulae, then so are $\bigcirc F$ and $F \mathcal{U} G$

Remark: Instead of $\bigcirc F$ in some books also XF is used.

Linear Time Logic

Semantics

- Transition systems $(S, \to, L)$
  (with the property that for every $s \in S$ there exists $s' \in S$ with $s \to s'$
  i.e. no state of the system can "deadlock"[a])

  Transition systems are also simply called models in what follows.

- Computation (execution, path) in a model $(S, \to, L)$
  infinite sequence of states $\pi = s_0, s_1, s_2, \ldots$ in S such that for each $i \geq 0$, $s_i \to s_{i+1}$.
  We write the path as $s_0 \to s_1 \to s_2 \to \ldots$.

[a] This is a technical convenience, and in fact it does not represent any real restriction on the systems we can model. If a system did deadlock, we could always add an extra state s_d representing deadlock, together with new transitions $s \to s_d$ for each s which was a deadlock in the old system, as well as $s_d \to s_d$.

Linear Time Logic

Consider the path $\pi = s_0 \rightarrow s_1 \rightarrow ....$

It represents a possible future of our system.

We write π^i for the suffix starting at s_i, e.g.,

$$\pi^3 = s_3 \rightarrow s_4 \rightarrow ....$$

Linear Time Logic

Semantics

Let $TS = (S, \rightarrow, L)$ be a model and $\pi = s_0 \rightarrow \ldots$ be a path in TS.

Whether π satisfies an LTL formula is defined by the satisfaction relation $\models$ as follows:

- $\pi \models \top$
- $\pi \not\models \bot$
- $\pi \models p$ iff $p \in L(s_0)$, if $p \in \Pi$
- $\pi \models \neg F$ iff $\pi \not\models F$
- $\pi \models F \wedge G$ iff $\pi \models F$ and $\pi \models G$
- $\pi \models F \vee G$ iff $\pi \models F$ or $\pi \models G$
- $\pi \models \bigcirc F$ iff $\pi^1 \models F$
- $\pi \models F \mathcal{U} G$ iff $\exists m \geq 0$ $s.t.$ $\pi^m \models G$ and $\forall k \in \{0, \ldots, m-1\} : \pi^k \models F$

Linear Time Logic

Alternative way of defining the semantics:

An LTL structure M is an infinite sequence $S_0 S_1 \ldots$ with $S_i \subseteq \Pi$ for all $i \geq 0$. We define satisfaction of LTL formulas in M at time points $n \in \mathbb{N}$ as follows:

- $M, n \models p$ iff $p \in S_n$, if $p \in \Pi$
- $M, n \models F \wedge G$ iff $M, n \models F$ and $M, n \models G$
- $M, n \models F \vee G$ iff $M, n \models F$ or $M, n \models G$
- $M, n \models \neg F$ iff $M, n \not\models F$
- $M, n \models \bigcirc F$ iff $M, n+1 \models F$
- $M, n \models F \mathcal{U} G$ iff $\exists m \geq n \; s.t. \; M, m \models G$ and $\forall k \in \{n, \ldots, m-1\} : M, k \models F$

Note that the time flow $(\mathbb{N}, <)$ is implicit.

Transition systems and LTL models

The connection between transition systems and LTL structures is as follows:

Every computation (evolution, path) of a transition system $s_0 \rightarrow s_1 \ldots$ gives rise to an LTL structure.

To see this, let $TS = (S, \rightarrow, L)$ be a transition system.

A computation $s_0, s_1, \ldots$ of TS induces an LTL structure $L(s_0)L(s_1)\ldots$

Such an LTL structure is called a trace of TS.

Abbreviations

- The future diamond

  $\Diamond\phi := \top\mathcal{U}\phi$ Sometimes denoted also $F\phi$

  $\pi \models \Diamond\phi$ iff $\exists m \geq 0 : \pi^m \models \phi$ $M, n \models \Diamond\phi$ iff $\exists m \geq n : M, m \models \phi$

- The future box

  $\Box\phi := \neg\Diamond\neg\phi$ Sometimes also denoted $G\phi$

  $\pi \models \Box\phi$ iff $\forall m \geq 0 : \pi^m \models \phi$ $M, n \models \Box\phi$ iff $\forall m \geq n : M, m \models \phi$

- The infinitely often operator

  $\Diamond^{\infty}\phi := \Box\Diamond\phi$

  $\pi \models \Diamond^{\infty}\phi$ iff $\{m \geq 0 \mid \pi^m \models \phi\}$ is infinite

  $M, n \models \Diamond^{\infty}\phi$ iff $\{m \geq n \mid M, m \models \phi\}$ is infinite

- The almost everywhere operator

  $\Box^{\infty}\phi := \Diamond\Box\phi$

  $\pi \models \Box^{\infty}\phi$ iff $\{m \geq 0 \mid \pi^m \not\models \phi\}$ is finite.

  $M, n \models \Box^{\infty}\phi$ iff $\{m \geq n \mid M, m \not\models \phi\}$ is finite.

Abbreviations

- The release operator
  $\phi \mathcal{R} \psi := \neg(\neg\phi \mathcal{U} \neg\psi)$

  $\pi \models \phi \mathcal{R} \psi$ iff ($\exists m \geq 0 : \pi^m \models \phi$ and $\forall k \leq m$: $\pi^k \models \psi$) or
  ($\forall k \geq 0 : \pi^k \models \psi$)

  $M, n \models \phi \mathcal{R} \psi$ iff ($\exists m \geq n : M, m \models \phi$ and $\forall k \leq m : M, m \models \psi$) or
  ($\forall k \geq m : M, k \models \psi$)

  Read as
  "ψ always holds unless released by ϕ" i.e.,
  "ψ holds permanently up to and including the first point where ϕ holds (such an ϕ-point need not exist at all)".

Abbreviations

- The strict until operator:
  $F\mathcal{U}^{<}G := \bigcirc(F\mathcal{U}G)$

  $\pi \models F\mathcal{U}^{<}G$ iff $\exists m > 0 : \pi^m \models G \wedge \forall k \in \{1, 2, \ldots, m-1\}, \pi^k \models F$

  $M, n \models F\mathcal{U}^{<}G$ iff $\exists m > n : M, m \models G \wedge \forall k \in \{n+1, ..., m-1\}, M, k \models F$

The difference between standard and strict until is that strict until requires G to happen in the strict future and that F needs not hold true of the current point.

Equivalence

We say that two LTL formulas F and G are (globally) equivalent
(written $F \equiv G$)
if, for all LTL structures M and $i \geq 0$, we have $M, i \models F$ iff $M, i \models G$.

equivalently:

if for all transition systems T and all paths π in T we have:
$\pi \models F$ iff $\pi \models G$.

Note that:

$\bigcirc F \equiv \bot \, \mathcal{U}^{<} F$ and

$F \mathcal{U} G \equiv G \vee (F \wedge (F \mathcal{U}^{<} G))$

Thus, an equally expressive version of LTL is obtained by using $\mathcal{U}^{<}$ as the only temporal operator.

This cannot be done with the standard until

Equivalence

Some useful equivalences that will be useful later on (exercise: prove them):

$\neg \bigcirc F \equiv \bigcirc \neg F$	(self-duality of next)
$\Diamond\Diamond F \equiv \Diamond F$	(idempotency of diamond)
$\bigcirc \Diamond F \equiv \Diamond \bigcirc F$	(commutation of next with Diamond)
$\Diamond\Diamond^{\infty} F \equiv \Diamond^{\infty} F \equiv \Diamond^{\infty} \Diamond F$	(absorption of diamonds by “infinitely ofte
$F \mathcal{U} G \equiv \neg(\neg F \mathcal{R} \neg G)$	(until and release are duals)
$F \mathcal{U} G \equiv G \vee (F \wedge \bigcirc(F \mathcal{U} G))$	(unfolding of until)
$F \mathcal{R} G \equiv (F \wedge G) \vee (G \wedge \bigcirc(F \mathcal{R} G))$	(unfolding of release)

Temporal Properties

A temporal property is a set of LTL structures
(those on which the property is true).

Thus, a temporal property P can be defined using an LTL formula F:

$$P = \{M \mid M, 0 \models F\}.$$

When given a transition system TS representing a reactive system and an LTL formula F representing a temporal property,

TS satisfies F if $M, 0 \models F$ for all traces M of TS.

In this case, we write $TS \models F$.

Typical properties of reactive systems that need to be checked during verification are safety properties, liveness properties, and fairness properties.

Safety properties

Intuitively, a safety property asserts that “nothing bad happens”

general form: Condition $\rightarrow \Box F_{\text{Safe}}$

Examples of safety properties:

- **Mutual Exclusion.** For the example:

$$\Box(\neg((A = 2) \wedge (B = 2)))$$

- **Freedom from Deadlocks:** At any time, some process should be enabled:

$$\Box(\mathit{enabled}_1 \vee \cdots \vee \mathit{enabled}_k)$$

- **Partial Correctness:** If F is satisfied when the program starts, then G will be satisfied if the program reaches a distinguished state:

$$F \rightarrow \Box(\text{Dist} \rightarrow G)$$

where Dist $\in \Pi$ marks the distinguished state.

Liveness properties

Intuitively, a liveness property asserts that "something good will happen"

Examples of liveness properties:

- **Guaranteed Accessibility.** For the example:

$$\Box(A = 1 \rightarrow \Diamond(A = 2)) \wedge \Box(B = 1 \rightarrow \Diamond(B = 2))$$

- **Responsiveness:** If a request is issued, it will eventually be granted:

$$\Box(\text{req} \rightarrow \Diamond\text{grant})$$

- **Total Correctness:** If F is satisfied when the program starts, then the program terminates in a distinguished state where G is satisfied:

$$\phi \rightarrow \Diamond(\text{Dist} \wedge G)$$

Note that, in contrast, partial correctness is a safety property.

Fairness properties

When modelling concurrent systems, it is usually important to make some fairness assumptions. Assume that there are k processes, that $\text{enabled}_i \in \Pi$ is true in a state s if process $\#i$ is enabled in s for execution, and that executed_i is true in a state s if process $\#i$ has been executed to reach s.

Examples of fairness properties

- **Unconditional Fairness:** Every process is executed infinitely often:

$$\bigwedge_{1 \leq i \leq k} \Diamond^{\infty} \text{executed}_i$$

  Unconditional fairness is appropriate when processes can (and should!) be executed and any time. This is not always the case.

Fairness properties

When modelling concurrent systems, it is usually important to make some fairness assumptions. Assume that there are k processes, that $\mathsf{enabled}_i \in \Pi$ is true in a state s if process $\#i$ is enabled in s for execution, and that $\mathsf{executed}_i$ is true in a state s if process $\#i$ has been executed to reach s.

Examples of fairness properties

- **Strong Fairness:** Every process enabled infinitely often is executed infinitely often:

$$\bigwedge_{1 \leq i \leq k} (\Diamond^{\infty}\mathsf{enabled}_i \to \Diamond^{\infty}\mathsf{executed}_i)$$

  Processes enabled only finitely often need not be guaranteed to be executed: they eventually and forever retract being enabled.

Fairness properties

When modelling concurrent systems, it is usually important to make some fairness assumptions. Assume that there are k processes, that $\mathsf{enabled}_i \in \Pi$ is true in a state s if process $\#i$ is enabled in s for execution, and that $\mathsf{executed}_i$ is true in a state s if process $\#i$ has been executed to reach s.

Examples of fairness properties

- **Weak Fairness:** Every process enabled almost everywhere is executed infinitely often.

  $$\bigwedge_{1 \leq i \leq k} (\Box^{\infty}\mathsf{enabled}_i \rightarrow \Diamond^{\infty}\mathsf{executed}_i)$$

  This means that a process cannot be enabled constantly in an infinite interval without being executed in this interval.

Semantics, Overview

TS transition system, $\pi = s_0 \to s_1 \to \dots$ path in TS.

$\pi \models F$ iff $L(s_0) \dots L(s_n), 0 \models F$

s state of TS.

$s \models F$ iff $(\forall \pi$ path starting in $s : \pi \models F)$

$$
\begin{aligned}
TS \models F \quad & \text{iff} \quad \pi \models F \text{ for all paths } \pi \\
& \text{iff} \quad s \models F \text{ for all states } s \text{ of } TS \\
& \text{iff} \quad M, 0 \models F \text{ for all traces } M \text{ of } TS
\end{aligned}
$$

Satisfiability

An LTL formula F is satisfiable
iff there exists a transition system TS and a path π such that $\pi \models F$
iff there exists a LTL structures M and $n \geq 0$ such that $M, n \models F$

Such a TS/structure is called a model of F.

In verification, satisfiability can be used to detect contradictory properties, i.e., properties that are satisfied by no computation of any reactive system.

Example: The following property is contradictory (unsatisfiable):

$$p \wedge \Box(p \rightarrow \bigcirc p) \wedge \Diamond \neg p$$

Satisfiability

When using LTL for verification, we are usually interested in whether a formula holds at point 0 of an LTL structure.

Lemma. Every satisfiable LTL formula F has a model M with $M, 0 \models F$.

Proof (Sketch)
Let $M, n \models F$, and let M' be the model obtained from M by dropping all time points $0, \ldots, n-1$. Thus, time point n in M is time point 0 in M'.

It is easy to prove by induction on the structure of G that, for all LTL formulas G and $i \geq 0$, we have $M', i \models G$ iff $M, n + i \models G$.

It follows that $M', 0 \models F$.

Semantics: Variants

Sometimes in the literature the models are of the form:

$TS = (S, \to, S_i, L)$, where S_i is a set of initial states.

Then:

$TS \models F$ iff $\pi \models F$ for all initial paths π

iff $s \models F$ for all initial states s of TS

Satisfiability

LTL satisfiability can be decided using automata on infinite words (Büchi automata).

Model checking

The LTL model checking problem is as follows: given a transition system $TS = (S, \rightarrow, S_i, L)$ and an LTL formula F, check whether $TS \models F$.

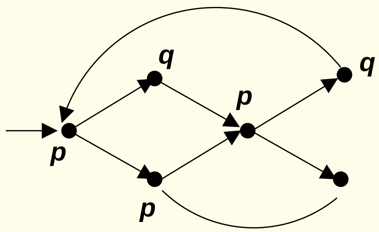
Model checking

The LTL model checking problem is as follows: given a transition system $TS = (S, \rightarrow, S_i, L)$ and an LTL formula F, check whether $TS \models F$.

Recall: this is the case iff

- all initial paths π of TS satisfy $\pi \models F$, iff
- for all initial states s of TS we have: $s \models F$.

Example:
The following transition system satisfies $\Box(q \rightarrow \bigcirc\bigcirc\bigcirc p)$.
It does not satisfy $\Box(p \rightarrow p\,\mathcal{U}\,q)$.



Connection to First-Order Logic

Another characterization of temporal properties that can be expressed in LTL is obtained by relating LTL to the monadic first-order theory of the natural numbers.

Let $FO^{<}$ denote the following first-order language:

- no function symbols and constants;
- binary predicate symbols: "suc" for successor, an order predicate $<$, and equality;
- countably infinite supply of unary predicates.

Connection to First-Order Logic

We may interpret formulas of $FO^{<}$ on LTL structures:

- quantification is over $\mathbb{N}$,
- the binary predicates are interpreted in the obvious way, and
- the unary predicates are identified with propositional variables.

Connection to First-Order Logic

We write $\phi(x_1, ..., x_n)$ to indicate that the variables in the $FO^<$ formula ϕ are $x_1, ..., x_n$.

For an $FO^<$ formula $\phi(x_1, ..., x_n)$, an LTL structure M, and $n_1, ..., n_k \in \mathbb{N}$, we write $M \models \phi[n_1, ..., n_k]$ if ϕ is true in M with variable x_i bound to value n_i, for $1 \leq i \leq k$.

Examples:

- For $\phi(x_1, x_2) = \neg p(x_1) \wedge p(x_2) \wedge \forall x_3.(x_1 < x_3 \rightarrow \neg q(x_3))$, we have $\varnothing\{p\} \ldots \{p\} \ldots \models \phi[0, 1]$.
- The following formula $\phi(x)$ expresses that there exists a future point that agrees with the current point (identified by the free variable) on the unary predicates $p_1, ..., p_n$:

$$\phi(x) = \exists y(x < y \wedge \bigwedge_{1 \leq i \leq n} (p_i(x) \leftrightarrow p_i(y)))$$

Connection to First-Order Logic

We say that an $FO^<$ formula $\phi(x)$ with exactly one free variable is equivalent to an LTL formula F if for all LTL models M and $n \in \mathbb{N}$ we have

$$M, n \models F \quad \text{iff} \quad M \models \phi[n].$$

Theorem: For every LTL formula F, there exists an equivalent $FO^<$ formula.

Proof The following translation $\mu : F_{LTL} \rightarrow FO^<$ takes LTL formulas F to equivalent $FO^<$ formulae:

$\mu(\top) = \top$; $\mu(\bot) = \bot$; $\mu(p)(x) = p(x)$ for every propositional variable p

$\mu(\neg F)(x) = \neg\, \mu(F)(x)$

$\mu(F \wedge G)(x) = \mu(F)(x) \wedge \mu(G)(x)$

$\mu(\bigcirc F)(x) = \exists y(suc(x, y) \wedge \mu(F)(y))$

$\mu(F \mathcal{U} G)(x) = \exists y(x \leq y \wedge \mu(G)(y) \wedge \forall z(x \leq z < y \rightarrow \mu(F)(z)))$

In the last two cases, variables y and z are newly introduced for every translation step.

Connection to First-Order Logic

What about the converse?

In general, are there $FO^{<}$ formulas $\phi(x)$ for which there is no equivalent LTL formula?

Connection to First-Order Logic

What about the converse?

In general, are there $FO^{<}$ formulas $\phi(x)$ for which there is no equivalent LTL formula?

Obviously there are: the formula $\exists y(y < x)$ states that there exists a previous time point – which cannot be expressed using only the future operators of LTL.

When we want to compare $FO^{<}$ with LTL, we should extend the latter with past-time temporal operators $\bigcirc^{-}$ and $\mathcal{S}$.

$M, n \models \bigcirc^{-} F$ iff $n > 0$ and $M, n-1 \models F$

$M, n \models F \mathcal{S} G$ iff $\exists m \leq n : M, m \models G$ and $M, k \models F$ for all $k \in \{m+1, ..., n\}$

Connection to First-Order Logic

This variant of LTL is called LTL with past operators (LTLP).

Connection to First-Order Logic

This variant of LTL is called LTL with past operators (LTLP).

Theorem (Kamp) For every $FO^<$ formula with one free variable, there exists an equivalent LTLP formula.

Proof. Out of the scope of this lecture.