Exercise 6.1 Prove the following equivalences of CTL formulae

(1) $\neg E\Diamond F \equiv A\Box\neg F$

$A\Box G$ is an abbreviation for the formula $\neg E\Diamond\neg G$.
(slides temporal-logic4.pdf, page 4).

Therefore $A\Box\neg F \overset{Def}{=} \neg E\Diamond\neg\neg F \equiv \neg E\Diamond F$.

(2) $E(F\mathcal{U}G) \equiv G \vee (F \wedge E\circ E(F\mathcal{U}G))$

We show that for every transition system $T=(S,\to,L)$ and every $s\in S$
$(T,s)\models E(F\mathcal{U}G)$ iff $(T,s)\models G\vee(F\wedge E\circ E(F\mathcal{U}G))$.

<u>Proof</u>. Let T be a transition system and s be a state of T

$(T,s)\models E(F\mathcal{U}G)$ iff (*) [there exists a computation $\pi = s_0 \to s_1 \to \cdots$ where $s_0 = s$ such that
$\exists m \geq 0$ with $(T,s_m)\models G$ and
$\forall k\in\{0,\dots m-1\}: (T,s_k)\models F$.

$(T,s)\models G\vee(F\wedge E\circ E(F\mathcal{U}G))$ iff [$(T,s)\models G$ or
$(T,s)\models F$ and $(T,s)\models E\circ E(F\mathcal{U}G)$.

iff [$(T,s)\models G$ or
$(T,s)\models F$ and there exists $s_1\in S$ w.s. $s\to s_1$
such that $(T,s_1)\models E(F\mathcal{U}G)$

iff [$(T,s)\models G$ or
$(T,s)\models F$ and there exists $s_1\in S$ w. $s\to s_1$
such that there exists $\pi = s_1\to s_2\to\cdots$
for which $\exists m\geq 1: (T,s_m)\models G$ and
$\forall k\in\{1\dots m-1\}: (T,s_k)\models F$.

It is easy to check that (*) and (**) are equivalent.

iff (**) [$(T,s)\models G$ or
there exists $s_1\in S$ s.t. there exists $\pi = s\to s_1\to s_2\to\cdots$
for which $\exists m\geq 1: (T,s_m)\models G$ and
$\forall k\in\{0,\dots m-1\}: (T,s_k)\models F$

(3) $E\Box F \equiv F \wedge E\circ E\Box F$.

Proof We show that for every transition system T and every state s of T
$(T,s) \models E\Box F$ iff $(T,s) \models F \wedge E\circ E\Box F$.

Let T be a trans. system and s be a state of T.

$(T,s) \models E\Box F$ iff [there exists a computation $\pi = s_0 \to s_1 \to \dots$ with $s_0 = s$
(*) such that $\forall m \geq 0$: $(T, s_m) \models F$.

$(T,s) \models F \wedge E\circ E\Box F$ iff [$(T,s) \models F$ and
$(T,s) \models E\circ E\Box F$

iff [$(T,s) \models F$ and
there exists s_1 with $s \to s_1$ such that
$(T,s_1) \models E\Box F$.

iff [$(T,s) \models F$ and
there exists s_1 with $s \to s_1$ and
there exists a computation $\pi = s_1 \to s_2 \to \dots$
such that $\forall m \geq 1$: $(T,s_m) \models F$

iff [there exists a computation $\pi = \overset{s}{\overset{\shortparallel}{s_0}} \to s_1 \to s_2 \to \dots$
such that $(T,s_0) = (T,s) \models F$ and
$\forall m \geq 1$: $(T,s_m) \models F$.

iff [there exists a computation $\pi = s_0 \to s_1 \to \dots$ with $s_0 = s$
(*) such that $\forall m \geq 0$: $(T,s_m) \models F$

iff $(T,s) \models E\Box F$.

(4) $\neg A(F \,\mathcal{U}\, G) \equiv E(\neg G \,\mathcal{U}\, (\neg F \wedge \neg G)) \vee E\square\neg G$. (3)

Proof. We show that for every transition system $T = (S, \to, L)$ and every $s \in S$:
$(T,s) \models \neg A(F \mathcal{U} G)$ iff $(T,s) \models E(\neg G \,\mathcal{U}\, (\neg F \wedge \neg G)) \vee E\square\neg G$.

Let T be a transition system and s be a state of T.

$(T,s) \models \neg A(F \mathcal{U} G)$ iff it is not true that (for all computations $\pi = s_0 \to s_1 \to \dots$ with $s_0 = s$ there is an $m \geq 0$ s.t. $(T, s_m) \models G$ and $\forall k \in \{0..m-1\}, (T, s_k) \models F$)

iff there exists a computation $\pi = s_0 \to s_1 \to \dots$ with $s_0 = s$ such that for all $m \geq 0$ ($(T, s_m) \models \neg G$ or $\exists k \in \{0..m-1\}$: $(T, s_k) \models \neg F$)

iff there exists a computation $\pi = s_0 \to s_1 \to \dots$ with $s_0 = s$ such that either for all $m \geq 0$ $(T, s_m) \models \neg G$
or there exists $m_0 \geq 0$ s.t. $(T, s_{m_0}) \models G$
and for all $m \geq 0$ ($(T, s_m) \models \neg G$ or $\exists k \in \{0 \dots m-1\}: (T, s_k) \models \neg F$)

$\exists x (A \vee B)$
$\equiv \exists x A \vee \exists x B$

iff there exists a computation $\pi = s_0 \to s_1 \to \dots$ with $s_0 = s$ such that for all $m \geq 0$ $(T, s_m) \models \neg G$

or

there exists a computation $\pi = s_0 \to s_1 \to \dots s_n \to \dots$ with $s_0 = s$ such that there exists $m_0 \geq 0$: $(T, s_{m_0}) \models G$
and for all $m \geq 0$ the following hold:
(1) if $m < m_0$ then $(T, s_m) \models \neg G$
(2) if $m = m_0$ then $\exists k \in \{0, \dots m-1\}$: $(T, s_k) \models \neg F$
(3) if $m > m_0$ then ($(T, s_m) \models \neg G$ or $\exists k \in \{0 \dots m-1\}: (T, s_k) \models \neg F$)

+ case distinction made clear.
If $\exists m_0$ with $(T, m_0) \models G$
$\Downarrow$ we choose the smallest such m_0
(then (1) holds.

iff there exists a computation $\pi = s_0 \to s_1 \to \dots$ with $s_0 = s$ such that for all $m \geq 0$ $(T, s_m) \models \neg G$
or
there exists a computation $\pi = s_0 \to s_1 \to \dots$ with $s_0 = s$ such that there exists m_0 with $(T, m_0) \models G$,
(*) $(T, m) \models \neg G$ for all $m \leq m_0$, $\exists k \in \{0, \dots m_0 - 1\}$ $(T, s_k) \models \neg F$.

$(T,s) \models E(\neg G \,\mathcal{U}\, (\neg F \wedge \neg G)) \vee E\Box\neg G$

iff $(T,s) \models E(\neg G \,\mathcal{U}\, (\neg F \wedge \neg G))$ or $(T,s) \models E\Box\neg G$

iff there exists a computation $\pi = s_0 \to s_1 \ldots s_n^{*}$ with $s_0 = s$ such that $\forall m \geq 0$ $(T, s_m) \models \neg G$.

or

(**) there exists a computation $\pi = s_0 \to s_1 \to \ldots$ with $s_0 = s$ such that $\exists n_0 \geq 0$ such that $(T, s_{n_0}) \models \neg F \wedge \neg G$ and $\forall k < n_0$: $(T, s_k) \models \neg G$.

we now show that (*) iff (**).

(*) $\Rightarrow$ (**) <u>Proof</u> If there exists a computation $\pi = s_0 \to s_1 \to \ldots$ with $s_0 = s$ such that for all $m \geq 0$, $(T, s_m) \models \neg G$ then (**) holds.

Assume now that there exists a computation $\pi = s_0 \to s_1 \to \ldots$ with $s_0 = s$ such that $\exists m_0$ with $(T, m_0) \models G$, $(T, m) \models \neg G$ for all $m < m_0$ and $\exists k \leq m_0 - 1$ such that $(T, s_k) \models \neg F$.

We choose n_0 to be the $\underline{k} \leq m_0 - 1$ for which $(T, s_k) \models \neg F$. Then $(T, s_k) \models \neg G$ because $k < m_0$, so $(T, n_0) \models \neg F \wedge \neg G$. In addition, we know that for all $k \leq n_0 \leq m_0 - 1$ we have $(T, s_k) \models \neg G$. Thus (**) holds also in this case.

(**) $\Rightarrow$ (*). <u>Proof</u> If there exists a computation $\pi = s_0 \to s_1 \to \ldots$ with $s_0 = s$ such that $\forall m \geq 0$, $(T, s_m) \models \neg G$, then (*) holds.

Assume now that this is not the case and there exists a computation $\pi = s_0 \to s_1 \to \ldots$ with $s_0 = s$ such that $(T, s_{n_0}) \models \neg F \wedge \neg G$ and $\exists n_0 \geq 0$ such that $(T, s_{n_0}) \models \neg F \wedge \neg G$ and $\forall k \leq n_0$: $(T, s_k) \models \neg G$.

Then: $\exists m_0 \geq 0$ such that $(T, s_{m_0}) \models G$, and $\forall m < m_0$: $(T, s_m) \models \neg G$.

$\exists k = n_0 \leq m_0$ such that $(T, s_k) \models \neg F$.

Thus (**) holds also in this case.