Exercise 4.1 Show that the following formulae are valid in PDL

(1) $[\alpha](A \wedge B) \leftrightarrow [\alpha]A \wedge [\alpha]B$

$[\alpha](A \wedge B) \leftrightarrow [\alpha]A \wedge [\alpha]B$ is valid iff for all PDL structures $K = (S, R(), I)$ and every $s \in S$ $(K,s) \models [\alpha](A \wedge B)$ iff $(K,s) \models [\alpha]A \wedge [\alpha]B$

Proof that holds.

Let K be a PDS structure and s a state of K.

$(K,s) \models [\alpha](A \wedge B)$ iff $\forall t \in S\,((s,t) \in R(\alpha) \rightarrow (K,t) \models A \wedge B))$

iff $\forall t \in S\,((s,t) \in R(\alpha) \rightarrow [(K,t) \models A \text{ and } (K,t) \models B])$

iff $\forall t \in S\,[((s,t) \in R(\alpha) \rightarrow (K,t) \models A)$ and $((s,t) \in R(\beta) \rightarrow (K,t) \models B)\,]$

iff $\forall t \in S\,[(s,t) \in R(\alpha) \rightarrow (K,t) \models A\,]$ and $\forall t \in S\,[(s,t) \in R(\alpha) \rightarrow (K,t) \models B\,]$

iff $(K,s) \models [\alpha]A$ and $(K,s) \models [\alpha]B$.

(2) $[\alpha;\beta]A \longleftrightarrow [\alpha][\beta]A$.

$[\alpha;\beta]A \leftrightarrow [\alpha][\beta]A$ is valid iff for all PDL structures $K = (S, R(), I)$ and all $s \in S$: $(K,s) \models [\alpha;\beta]A$ iff $(K,s) \models [\alpha][\beta]A$

Proof that this is the case:

Let K be a PDL structure and s a state of K.

$(K,s) \models [\alpha;\beta]A$ iff $\forall t\,(\underbrace{(s,t) \in R(\alpha;\beta)} \rightarrow (K,t) \models A)$

iff $\forall t\,((\exists u: (s,u) \in R(\alpha) \wedge (u,t) \in R(\beta)) \rightarrow (K,t) \models A)$

iff $\forall t\,(\underbrace{(\neg \exists u((s,u) \in R(\alpha) \wedge (u,t) \in R(\beta))}_{\equiv\ \forall u((s,u) \notin R(\alpha) \vee (u,t) \notin R(\beta))} \vee (K,t) \models A)$

iff $\boxed{\forall t \forall u\,((s,u) \notin R(\alpha) \vee (u,t) \notin R(\beta) \vee (K,t) \models A)}$ (*)

$(K,s) \models [\alpha][\beta]A$ iff $\forall u((s,u) \in R(\alpha) \rightarrow (K,u) \models [\beta]A)$

iff $\forall u((s,u) \in R(\alpha) \rightarrow \forall t((u,t) \in R(\beta) \rightarrow (K,t) \models A)$

iff $\forall u((s,u) \notin R(\alpha) \vee \forall t((u,t) \notin R(\beta) \vee (K,t) \models A))$

iff $\boxed{\forall u \forall t\,((s,u) \notin R(\alpha) \vee (u,t) \notin R(\beta) \vee (K,t) \models A)}$ (**)

It is easy to see that (*) and (**) are equivalent.

(3) $[\alpha \cup \beta]A \leftrightarrow [\alpha]A \wedge [\beta]A$

We show that for every PDL structure $K = (S, R(), I)$ and every $s \in S$
$(K,s) \models [\alpha \cup \beta]A$ iff $(K,s) \models [\alpha]A \wedge [\beta]A$.

Proof:

$(K,s) \models [\alpha \cup \beta]A$ iff $\forall t \in S$ with $(s,t) \in R(\alpha \cup \beta)$: $(K,t) \models A$
iff $\forall t \in S$ with $(s,t) \in R(\alpha) \cup R(\beta)$: $(K,t) \models A$
iff $\forall t \in S$ with $(s,t) \in R(\alpha)$: $(K,t) \models A$
and
$\forall t \in S$ with $(s,t) \in R(\beta)$: $(K,t) \models A$.

iff $(K,s) \models [\alpha]A$ and
$(K,s) \models [\beta]A$.
iff $(K,s) \models [\alpha]A \wedge [\beta]A$

(4) $[A?]B \leftrightarrow (A \rightarrow B)$.

We show that for every PDL structure K and every state s of K
$(K,s) \models [A?]B$ iff $(K,s) \models (A \rightarrow B)$.

Proof:

$(K,s) \models [A?]B$ iff $\forall t \in S \; \big((s,t) \in R(A?) \rightarrow (K,t) \models B \big)$
iff $\forall t \in S \; \big((s = t \text{ and } (K,s) \models A) \rightarrow (K,t) \models B \big)$ (*)

$(K,s) \models A \rightarrow B$ iff $(K,s) \models A \rightarrow (K,s) \models B$. (**)

We prove that (*) and (**) are equivalent.

(*) $\Rightarrow$ (**)
Assume (*) holds. Assume $(K,s) \models A$. In (*) we take $t = s$. Then the premises of (*) hold, hence also the conclusion of (*) holds. Thus $(K,t) \models B$, so as $t = s$ we have $(K,s) \models B$. QED

(**) $\Rightarrow$ (*)
Assume (**). Let $t \in S$. Assume $s = t$ and $(K,s) \models A$. By (**) $(K,s) \models B$
As $s = t$, it follows that $(K,t) \models B$. QED

⑤ $[\alpha^*]A \leftrightarrow A \wedge [\alpha][\alpha^*]A$.

We show that for every PDL structure K and every state s of K
$(K,s) \models [\alpha^*]A$ iff $(K,s) \models A \wedge [\alpha][\alpha^*]A$.

Proof:

$(K,s) \models [\alpha^*]A$ iff $\forall t \, \big((s,t) \in R(\alpha^*) \to (K,t) \models A \big)$

iff $\forall t \, \left(\left[\begin{array}{l} \exists n \in N \\ \exists t_0 t_1 \dots t_n : t_i R(\alpha) t_{i+1} \\ t_0 = s,\ t_n = t \end{array} \right] \to (K,t) \models A \right)$

iff $\left[\forall n \in N \ \forall t_0 t_1 \dots t_n \left(\begin{array}{l} t_0 = s \text{ and} \\ t_i R(\alpha) t_{i+1} \\ i = 0..n-1 \end{array} \right) \to (K,t_n) \models A \right]$

iff $\left[\begin{array}{l} \text{for } n = 0 :\ (K,t_0) \models A \text{ where } t_0 = s \quad (*) \\ \text{and} \\ \text{for } n \geq 1 \ \forall t_1 \dots t_n \left(\begin{array}{l} s R(\alpha) t_1 \text{ and} \\ t_i R(\alpha) t_{i+1} \\ i = 1..n-1 \end{array} \right) \to (K,t_n) \models A \quad (**) \end{array} \right.$

$(K,s) \models A \wedge [\alpha][\alpha^*]A$ iff $\left\{ \begin{array}{l} (K,s) \models A \text{ and} \\ (K,s) \models [\alpha][\alpha^*]A \end{array} \right.$

iff $\left[\begin{array}{l} (K,s) \models A \text{ and} \\ \forall t_1 \big(s R(\alpha) t_1 \to (K,t_1) \models [\alpha^*]A \big) \end{array} \right.$

iff $\left[\begin{array}{l} (K,s) \models A \text{ and} \\ \forall t_1 \left(s R(\alpha) t_1 \to \forall n \geq 1 \ \forall t_1 \dots t_n \left(\begin{array}{l} t_i R(\alpha) t_{i+1} \\ i = 1..n-1 \end{array} \to (K,t_n) \models A \right) \right) \end{array} \right.$

iff $\left[\begin{array}{l} (K,s) \models A \text{ and} \\ \forall n \geq 1 \ \forall t_1, \dots, t_n \left(s R(\alpha) t_1 \to \left(\begin{array}{l} t_i R(\alpha) t_{i+1} \\ i = 1..n-1 \end{array} \to (K,t_n) \models A \right) \right) \end{array} \right.$

iff $\left[\begin{array}{l} (K,s) \models A \text{ and} \quad (*) \\ \forall n \geq 1 \ \forall t_1 \dots t_n \left(\begin{array}{l} s R(\alpha) t_1 \text{ and} \\ t_i R(\alpha) t_{i+1} \\ i = 1..n-1 \end{array} \right) \to (K,t_n) \models A \quad (**) \end{array} \right.$

It is easy to see that $((*)$ and $(*))$ and $((**)$ and $(**))$ are equivalent.

⑥ $[\alpha^*](A \to [\alpha]A) \to (A \to [\alpha^*]A)$

We show that for every PDL structure K and every state s of K:
If $(K,s) \models [\alpha^*](A \to [\alpha]A)$ then $(K,s) \models A \to [\alpha^*]A$.

Proof:

Assumption: $(K,s) \models [\alpha^*](A \to [\alpha]A)$

i.e. $\forall t \in S \; (\underbrace{s R(\alpha^*) t} \to (K,t) \models (A \to [\alpha]A))$

i.e. $\left[\begin{array}{l} \forall t \in S \\ \forall n \in \mathbb{N}, \forall t_0 t_1 \dots t_n \left(\left(\begin{array}{l} s = t_0 \wedge t = t_n \\ \text{and} \\ t_i R(\alpha) t_{i+1} \\ \text{for all } i = 0 \dots n-1 \end{array} \right) \to (K,t) \models A \to [\alpha]A \right) \end{array} \right.$

To prove: $(K,s) \models A \to [\alpha^*]A$,

i.e. If $(K,s) \models A$ then $(K,s) \models [\alpha^*]A$.

Proof

Assume $(K,s) \models A$.

We show that $(K,s) \models [\alpha^*]A$ by proving that

for all $n \geq 0$ the following property holds:

$p(n)$: for all $t_0 t_1 \dots t_n \in S$ such that $t_0 = s$ and $(t_i, t_{i+1}) \in R(\alpha)$ for all $i = 0, \dots n-1$ we have $(K, t_n) \models A$.

We prove this by induction:

Ind. basis: $n=0$ Then: $t_n = s$ and we know that $(K, \underset{s}{t_n}) \models A$

Ind hypothesis: Assume $p(n)$ holds.

Ind. step: Prove that $p(n+1)$ holds.
Let $t_0 t_1 \dots t_n t_{n+1} \in S$ such that $t_0 = s$ and
$(t_i, t_{i+1}) \in R(\alpha)$ for all $i = 0, 1, \dots n$.

- By the ind. hypothesis we know that $(K, t_n) \models A$.
- On the other side, $(t_0, t_n) \in R(\alpha^*)$. By the assumption, we know that $(K, t_n) \models A \to [\alpha]A$.

It follows that $(K, t_n) \models [\alpha]A$, so $(K, t_{n+1}) \models A$. QED