Exercise 5.1

$\Sigma = (\Omega, \Pi)$ with $\Omega = \{b/0, f/1\}$ and $\Pi = \{p/1\}$.

$\boxed{\text{(1) How many Herbrand interpretations over } \Sigma \text{ exist?}}$

Solution: $\mathcal{H} = (U_{\mathcal{H}}, \{f_{\mathcal{H}}, b_{\mathcal{H}}\}, \{p_{\mathcal{H}}\})$ is a Herbrand interpretation over Σ if
(1) $U_{\mathcal{H}} = T_\Sigma = \{b, f(b), f^2(b), \ldots\}$
(2) $f_{\mathcal{H}} : T_\Sigma \to T_\Sigma$ is defined by $f_{\mathcal{H}}(f^n(b)) = f(f^n(b)) = f^{n+1}(b)$.
(3) $b_{\mathcal{H}} = b \in T_\Sigma$
(3) $p_{\mathcal{H}} \subseteq T_\Sigma$

The universe and the interpretation of the function symbols are fixed. Therefore, there are as many Herbrand interpretations over Σ as subsets of T_Σ.

$$|\{\mathcal{H} \mid \mathcal{H} \text{ Herbrand interpretation over } \Sigma\}| = |\mathcal{P}(T_\Sigma)| \underset{\uparrow}{=} |\mathcal{P}(\mathbb{N})|$$

because there is a bijection from $\mathbb{N}$ to T_Σ: $i : \mathbb{N} \to T_\Sigma$, $i(n) = f^n(b)$.

Since $\mathcal{P}(\mathbb{N})$ is not countable, the ~~set of~~ all Herbrand interpretations over Σ is not countable.

[It is known, in fact, that $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$. The proof is given in a different file]

$\boxed{\text{(2) Consider } F := p(f(f(b))) \wedge \forall x\, (p(x) \to p(f(x))). \text{ How many different Herbrand models does } F \text{ have?}}$

Let $\mathcal{A} = (T_\Sigma, \{f_{\mathcal{A}} : T_\Sigma \to T_\Sigma,\ f_{\mathcal{A}}(f^n(b)) = f^{n+1}(b),\ b_{\mathcal{A}} = b \in T_\Sigma\},\ p_{\mathcal{A}} \subseteq T_\Sigma)$ be a Herbrand interpretation such that $\mathcal{A} \models F$.

Then $\mathcal{A} \models p(f(f(b)))$, i.e. $p_{\mathcal{A}}(f(f(b))) = 1$ (i.e. $f(f(b)) \in p_{\mathcal{A}}$) (*)

$\mathcal{A} \models \forall x\, (p(x) \to p(f(x)))$, i.e. $\forall \beta : X \to T_\Sigma$, $\mathcal{A}(\beta)(\forall x\, p(x) \to p(f(x))) = 1$

i.e. $\forall \beta : X \to T_\Sigma$ $\min_{t \in T_\Sigma} \mathcal{A}(\beta[x \mapsto t])(p(x) \to p(f(x))) = 1$

i.e. $\forall \beta : X \to T_\Sigma$ and $\forall t \in T_\Sigma$: $\mathcal{A}(\beta[x \mapsto t])(p(x) \to p(f(x))) = 1$

i.e. $\forall \beta : X \to T_\Sigma$ and $\forall t \in T_\Sigma$: $p_{\mathcal{A}}(t) \to p_{\mathcal{A}}(f(t)) = 1$

i.e. $\forall \beta : X \to T_\Sigma$ and $\forall t = f^n(b)$: $p_{\mathcal{A}}(f^n(b)) \to p_{\mathcal{A}}(f^{n+1}(b)) = 1$ (**)

Hence, if $\mathcal{A} \models F$ then

(i) $p_{\mathcal{A}}(f(f(b))) = 1$ (*)
(ii) $(p_{\mathcal{A}}(f(f(b))) \to p_{\mathcal{A}}(f(f(f(b))))) = 1$ (**) | hence (iii) $p_{\mathcal{A}}(f^3(b)) = 1$
(iii) $(p_{\mathcal{A}}(f^3(b)) \to p_{\mathcal{A}}(f^4(b))) = 1$ (**) | hence (iv) $p_{\mathcal{A}}(f^4(b)) = 1$

....

one can prove by induction that $p_{\mathcal{A}}(f^n(b)) = 1$ for all $n \geq 2$. Nothing is known about $p_{\mathcal{A}}(b)$, $p_{\mathcal{A}}(f(b))$

We showed that if $\mathcal{A}$ is a Herbrand interpretation such that $\mathcal{A} \models F$, then $p_{\mathcal{A}}(f^n(b)) = 1$ for all $n \geq 2$.

Case 1 $p_{\mathcal{A}}(b) = 1$. Then by (**) $p_{\mathcal{A}}(f(b)) = 1$, hence $p_{\mathcal{A}} = \{b, f(b), f^2(b), \dots\} = T_\Sigma$.

Case 2 $p_{\mathcal{A}}(b) = 0$. We distinguish two subcases: Case 2.1: $p_{\mathcal{A}}(b) = 1$. Then $p_{\mathcal{A}} = \{f(b), f^2(b), \dots\} = T_\Sigma \setminus \{b\}$

Case 2.2 $p_{\mathcal{A}}(b) = 0$. Then $p_{\mathcal{A}} = \{f^2(b), f^3(b), \dots\} = T_\Sigma \setminus \{b, f(b)\}$.

There are 3 interpretations:

$\mathcal{A}_1 = (T_\Sigma, \{f_{\mathcal{A}_1}, b_{\mathcal{A}_1}\}, p_{\mathcal{A}_1} = T_\Sigma)$

$\mathcal{A}_2 = (T_\Sigma, \{f_{\mathcal{A}_2}, b_{\mathcal{A}_2}\}, p_{\mathcal{A}_2} = \{f^n(b) \mid n \geq 1\})$

$\mathcal{A}_3 = (\underbrace{T_\Sigma, \{f_{\mathcal{A}_3}, b_{\mathcal{A}_3}\}}_{\text{standard, in any Herbrand int.}}, p_{\mathcal{A}_3} = \{f^n(b) \mid n \geq 2\})$.

(3) Every Herbrand interpretation which is a model of F is also a model of $G = \forall x\ p(f(f(x)))$.
Give an example of an algebra that is a model of F but not of G.

Solution: Let $\mathcal{A} = (U_{\mathcal{A}}, \{f_{\mathcal{A}}, b_{\mathcal{A}}\}, p_{\mathcal{A}})$ where

$U_{\mathcal{A}} = \{a, b, f(a), f(b), f^2(a), f^2(b), \dots\}$
$= \{f^n(a) \mid n \geq 0\} \cup \{f^n(b) \mid n \geq 0\}$

$f_{\mathcal{A}} : U_{\mathcal{A}} \to U_{\mathcal{A}}$ defined by $f_{\mathcal{A}}(f^n(a)) = f^{n+1}(a)$
$f_{\mathcal{A}}(f^n(b)) = f^{n+1}(b)$.

$b_{\mathcal{A}} \in U_{\mathcal{A}}$ $b_{\mathcal{A}} = b$.

$p_{\mathcal{A}} = \{f^n(b) \mid n \geq 2\}$.

Then $\mathcal{A} \models F$ but $\mathcal{A} \not\models G$:

In fact, for all $\beta: X \to U_{\mathcal{A}}$: $\mathcal{A}(\beta)(G) = 0$:

$\mathcal{A}(\beta)(G) = \mathcal{A}(\beta)(\forall x\ p(f(f(x)))) = \min_{t \in U_{\mathcal{A}}} p_{\mathcal{A}}(f(f(t))) = 0$

because for $t = a$ we have $p_{\mathcal{A}}(f^2(a)) = 0$

(4) Let $\mathcal{A}$ be a Herbrand interpretation over Σ and $\sim$ the binary relation on T_Σ defined by:
$t_1 \sim t_2$ iff $(\forall x\ f^3(x) \simeq x) \models t_1 \approx t_2$

a) Is $\sim$ a congruence relation on $\mathcal{A}$?

Solution

$\sim$ is a congruence relation on $\mathcal{A}$ iff $\left\{ \begin{array}{l} \sim \subseteq T_\Sigma \times T_\Sigma \text{ is an equivalence relation} \\ \text{and it is "compatible" with } f, p. \end{array} \right.$

I $\sim$ is an equivalence relation

(i) $\sim$ is reflexive, i.e. $\forall t \in T_\Sigma$, $t \sim t$.

Proof: we show that $\forall x\ f^3(x) \simeq x \models t \approx t$, i.e. that for all $\mathcal{A}'$ Σ-structure, $\beta : X \to U_{\mathcal{A}'}$, if $\mathcal{A}'(\beta)(\forall x (f^3(x) \simeq x)) = 1$ then $\underbrace{\mathcal{A}'(\beta)(t) = \mathcal{A}'(\beta)(t)}_{\text{always equal}}$.

always true; q.e.d.

(ii) $\sim$ is symmetric, i.e. $\forall t \in T_\Sigma$, if $t_1 \sim t_2$ then $t_2 \sim t_1$.

Proof Assume $t_1 \sim t_2$. We show that $t_2 \sim t_1$, i.e. $(\forall x f^3(x) \simeq x) \models t_2 \approx t_1$.
Let $\mathcal{A}'$ be a Σ-structure and $\beta : X \to U_{\mathcal{A}'}$ such that $\mathcal{A}'(\beta)(\forall x\ f^3(x) = x) = 1$.
Since we assumed $t_1 \sim t_2$, i.e. $(\forall x\ f^3(x) \simeq x) \models t_1 \approx t_2$, we know that $\mathcal{A}'(\beta)(t_1) = \mathcal{A}'(\beta)(t_2)$. But then $\mathcal{A}'(\beta)(t_2) = \mathcal{A}'(\beta)(t_1)$.

(iii) $\sim$ is transitive, i.e. $\forall t \in T_\Sigma$ if $t_1 \sim t_2$ and $t_2 \sim t_3$ then $t_1 \sim t_3$.

Proof Assume $t_1 \sim t_2$ and $t_2 \sim t_3$.
We prove that $t_1 \sim t_3$ i.e. $(\forall x f^3(x) \simeq x) \models t_1 \approx t_3$.
Let $\mathcal{A}'$ be a Σ-structure and $\beta : X \to U_{\mathcal{A}'}$ be such that $\mathcal{A}'(\beta)(\forall x\ f^3(x) = x) = 1$
- Since we assumed that $t_1 \sim t_2$, i.e. $(\forall x\ f^3(x) = x) \models t_1 \simeq t_2$, we know that $\mathcal{A}'(\beta)(t_1) = \mathcal{A}'(\beta)(t_2)$.
- Since we assumed that $t_2 \sim t_3$ i.e. $(\forall x\ f^3(x) = x) \models t_2 \simeq t_3$, we know that $\mathcal{A}'(\beta)(t_2) = \mathcal{A}'(\beta)(t_3)$.

It follows that $\mathcal{A}'(\beta)(t_1) = \mathcal{A}'(\beta)(t_2) = \mathcal{A}'(\beta)(t_3)$, hence $\mathcal{A}'(\beta)(t_1) = \mathcal{A}'(\beta)(t_3)$.
We thus showed that $(\forall x\ f^3(x) \simeq x) \models t_1 \simeq t_3$, i.e. that $t_1 \sim t_3$.

(iv) $\sim$ is "compatible" with f, i.e. $\forall t_1, t_2$ if $t_1 \sim t_2$ then $f(t_1) \sim f(t_2)$

Proof: Assume $t_1 \sim t_2$, i.e. $(\forall x\, f^3(x) = x) \models t_1 \approx t_2$ (*)
we show that $f(t_1) \sim f(t_2)$, i.e. $(\forall x\, f^3(x) = x) \models f(t_1) \approx f(t_2)$,
Let $\mathcal{A}'$ be a Σ-structure, $\beta : X \to U_{\mathcal{A}'}$ be st. $\mathcal{A}', \beta \models \forall x\, f^3(x) = x$.
By (*) we know that $\mathcal{A}'(\beta)(t_1) = \mathcal{A}'(\beta)(t_2)$.
Then $\mathcal{A}'(\beta)(f(t_1)) = f_{\mathcal{A}'}(\mathcal{A}'(\beta)(t_1)) = f_{\mathcal{A}'}(\mathcal{A}'(\beta)(t_2)) = \mathcal{A}'(\beta)(f(t_2))$
This shows that $\forall x\, f^3(x) \approx x \models f(t_1) \approx f(t_2)$, hence that $f(t_1) \sim f(t_2)$

(v) we study compatibility with p.
We check whether $\forall t_1, t_2 \in T_\Sigma$ if $t_1 \sim t_2$ and $p_{\mathcal{A}}(t_1) = 1$
then $p_{\mathcal{A}}(t_2) \overset{?}{=} 1$

This is not always the case:
We can take for instance the Herbrand interpretation

$$\mathcal{A} = (T_\Sigma, \underbrace{\{f_{\mathcal{A}}, b_{\mathcal{A}}\}}_{\text{usual definition}}, P_{\mathcal{A}} = \{f^3(b)\})$$

i.e. only $P_{\mathcal{A}}(f^3(b))$ is true,

For $t_2 = b$ and $t_1 = f^3(b)$ we have:

$t_1 \sim t_2$ and $p_{\mathcal{A}}(t_1) = 1$, but $P_{\mathcal{A}}(t_2) = 0$..

$\sim$ is not a congruence on all Herbrand interpretations.
it is a congruence only for those Herbrand interpretations:

$\mathcal{A} = (T_\Sigma, \underbrace{\{f_{\mathcal{A}}, b_{\mathcal{A}}\}}_{\text{usual def.}}, P_{\mathcal{A}})$ where $P_{\mathcal{A}}(b) = P_{\mathcal{A}}(f^{3k}(b))$ for all k
and $P_{\mathcal{A}}(f(b)) = P_{\mathcal{A}}(f^{3k+1}(b))$ for all k
and. $P_{\mathcal{A}}(f^2(b)) = P_{\mathcal{A}}(f^{3k+2}(b))$ for all k.

Assume that $\mathcal{A} = (T_\Sigma, \{f_\mathcal{A}, b_\mathcal{A}\}, P_\mathcal{A})$
has the property that

$$P_\mathcal{A}(b) = P_\mathcal{A}(f^{3k}(b))$$
$$P_\mathcal{A}(f(b)) = P_\mathcal{A}(f^{3k+1}(b))$$
$$P_\mathcal{A}(f^2(b)) = P_\mathcal{A}(f^{3k+2}(b)) \qquad \text{for all } k \in \mathbb{N}.$$

we can build the quotient

$$\mathcal{A}/_\sim = (T_\Sigma/_\sim, \{f_{\mathcal{A}/\sim}, b_{\mathcal{A}/\sim}\}, P_{\mathcal{A}/\sim}) \text{ as follows:}$$

$$T_\Sigma/_\sim = \{ [b], [f(b)], [f^2(b)] \}$$

where
$$[b] = \{ t \in T_\Sigma \mid t \sim b \} = \{ b, f^3(b), \dots f^{3k}(b), \dots \}$$
$$[f(b)] = \{ t \in T_\Sigma \mid t \sim f(b) \} = \{ f(b), f^4(b), \dots f^{3k+1}(b), \dots \}$$
$$[f^2(b)] = \{ t \in T_\Sigma \mid t \sim f^2(b) \} = \{ f^2(b), f^5(b), \dots f^{3k+2}(b), \dots \}$$

are the equivalence classes wrt. $\sim$.

$$f_{\mathcal{A}/\sim}([b]) = [f(b)]$$
$$f_{\mathcal{A}/\sim}([f(b)]) = [f^2(b)]$$
$$f_{\mathcal{A}/\sim}([f^2(b)]) = [b].$$

$$P_{\mathcal{A}/\sim}([b]) := P_\mathcal{A}(b)$$ (well-defined because congruence)
$$P_{\mathcal{A}/\sim}([f(b)]) := P_\mathcal{A}(f(b))$$
$$P_{\mathcal{A}/\sim}([f^2(b)]) := P_\mathcal{A}(f(f(b))).$$