Exercise **9.2** Prove the following equivalences of CTL formulae

(1) $\neg E \Diamond F \equiv A \Box \neg F$

AaG is an abbreviation for the formula $\neg E \Diamond \neg G$.
(slides temporal-logic4.pdf, page 4).

Therefore $A \Box \neg F \overset{Def}{=} \neg E \Diamond \neg \neg F \equiv \neg E \Diamond F$.

(2) $E(F \cup G) \equiv G \vee (F \wedge E \circ E(F \cup G))$

we show that for every transition system $T = (S, \rightarrow, L)$ and every $s \in S$

$(T, s) \models E(F \cup G)$ iff $(T, s) \models G \vee (F \wedge E \circ E(F \cup G))$.

Proof. Let T be a transition system and s be a state of T

$(T, s) \models E(F \cup G)$ iff (*) $\begin{cases} \text{there exists a computation } \pi = s_0 \rightarrow s_1 \rightarrow \cdots \\ \text{where } s_0 = s \text{ such that} \\ \quad \exists m \geq 0 \text{ with } (T, s_m) \models G \text{ and} \\ \quad \forall k \in \{0, \ldots m-1\} : (T, s_k) \models F. \end{cases}$

$(T, s) \models G \vee (F \wedge E \circ E(F \cup G))$ iff $\begin{cases} (T, s) \models G \text{ or} \\ (T, s) \models F \text{ and } (T, s) \models E \circ E(F \cup G). \end{cases}$

iff $\begin{cases} (T, s) \models G \text{ or} \\ (T, s) \models F \text{ and there exists } s_1 \in S \text{ w. } s \rightarrow s_1 \\ \text{such that } (T, s_1) \models E(F \cup G) \end{cases}$

iff $\begin{cases} (T, s) \models G \text{ or} \\ (T, s) \models F \text{ and there exists } s_1 \in S \text{ w. } s \rightarrow s_1 \\ \text{such that there exists } \pi = s_1 \rightarrow s_2 \rightarrow \cdots \\ \text{for which } \exists m \geq 1 : (T, s_m) \models G \text{ and} \\ \forall k \in \{1 \cdots m-1\} : (T, s_k) \models F. \end{cases}$

It is easy to check that (*) and (**) are equivalent.

iff (**) $\begin{cases} (T, s) \models G \text{ or} \\ \text{there exists } s_1 \in S \text{ s.t. there exists } \pi = s \rightarrow s_1 \rightarrow s_2 \rightarrow \cdots \\ \text{for which } \exists m \geq 1 : (T, s_m) \models G \text{ and} \\ \forall k \in \{0, \ldots m-1\} : (T, s_k) \models F \end{cases}$

(3) $E\square F \equiv F \wedge E \circ E \square F.$

Proof We show that for every transition system T and every state s of T

$$(T,s) \models E\square F \quad \text{iff} \quad (T,s) \models F \wedge E \circ E \square F.$$

Let T be a trans. system and s be a state of T.

$(T,s) \models E\square F$ iff $\underset{(*)}{\Big[}$ there exists a computation $\pi = s_0 \to s_1 \to \dots$ with $s_0 = s$
such that $\forall u \geq 0: (T, s_u) \models F$.

$(T,s) \models F \wedge E \circ E \square F$ iff $\Big[$ $(T,s) \models F$ and.
$(T,s) \models E \circ E \square F$

iff $\Big[$ $(T,s) \models F$ and
there exists s_1 with $s \to s_1$ such that
$(T, s_1) \models E \square F$.

iff $\Big[$ $(T,s) \models F$ and
there exists s_1 with $s \to s_1$ and
there exists a computation $\pi = s_1 \to s_2 \to \dots$
such that $\forall u \geq 1 : (T, s_u) \models F$

iff $\Big[$ there exists a computation $\pi = \overset{s}{\overset{\shortparallel}{s_0}} \to s_1 \to s_2 \to \dots$
such that $(T, s_0) = (T, s) \models F$ and.
$\forall u \geq 1 : (T, s_u) \models F$.

iff $\underset{(*)}{\Big[}$ there exists a computation $\pi = s_0 \to s_1 \to \dots$ with $s_0 = s$
such that $\forall u \geq 0 : (T, s_u) \models F$

iff $(T,s) \models E\square F.$

(4) $\neg A(F u G) \equiv E(\neg G \; u \; (\neg F \wedge \neg G)) \vee E \Box \neg G.$

Proof. We show that for every transition system $T = (S, \rightarrow, L)$ and every $s \in S$:

$(T,s) \models \neg A(F u G)$ iff $(T,s) \models E(\neg G \; u \; (\neg F \wedge \neg G)) \vee E \Box \neg G.$

Let T be a transition system and s be a state of T.

$(T,s) \models \neg A(F u G)$ iff

it is not true that (for all computations $\pi = s_0 \rightarrow s_1 \rightarrow \cdots$
with $s_0 = s$ there is an $m \geq 0$
s.t. $(T, s_m) \models G$ and $\# k \in \{0 \ldots m-1\}, (T, s_k) \models F$)

iff

there exists a computation $\pi = s_0 \rightarrow s_1 \rightarrow \cdots$ with $s_0 = s$
such that for all $m \geq 0$ ($(T, s_m) \models \neg G$ or $\exists k \in \{0 \ldots m-1\}$:
$(T, s_k) \models \neg F$)

iff

there exists a computation $\pi = s_0 \rightarrow s_1 \rightarrow \cdots$ with $s_0 = s$
such that either for all $m \geq 0$ $(T, s_m) \models \neg G$
or there exists $m_0 \geq 0$ st. $(T, s_{m_0}) \models G$
and for all $m \geq 0$ ($(T, s_m) \models \neg G$ or
$\exists k \in \{0 \ldots m-1\}: (T, s_k) \models \neg F$)

$\exists x (A \vee B)$
$\equiv \exists x A \vee \exists x B$

iff

there exists a computation $\pi = s_0 \rightarrow s_1 \rightarrow \cdots$ with $s_0 = s$
such that for all $m \geq 0$ $(T, s_m) \models \neg G$

or

there exists a computation $\pi = s_0 \rightarrow s_1 \rightarrow \cdots s_m \rightarrow \cdots$ with $s_0 = s$
such that there exists $m_0 \geq 0: (T, s_{m_0}) \models G$
and for all $m \geq 0$ the following hold :
(1) If $m < m_0$ then $(T, s_m) \models \neg G$
(2) If $m = m_0$ then $\exists k \in \{0, \ldots m-1\}: (T, s_k) \models \neg F$
(3) If $m > m_0$ then ($(T, s_m) \models \neg G$ or
$\exists k \in \{0 \ldots m-1\}: (T, s_k) \models \neg F$)

+ case distinction
made clear.

If $\exists m_0$ with $(T, m_0) \models G$
we choose
the smallest such m_0
(then (1) holds).

iff

there exists a computation $\pi = s_0 \rightarrow s_1 \rightarrow \cdots$ with $s_0 = s$
such that for all $m \geq 0$ $(T, s_m) \models \neg G$
or
there exists a computation $\pi = s_0 \rightarrow s_1 \rightarrow \cdots$ with $s_0 = s$
such that there exists m_0 with $(T, m_0) \models G$,
$(T, m) \models \neg G$ for all $m \leq m_0$, $\exists k \in \{0, \ldots m_0 - 1\}$ $(T, s_k) \models \neg F$.

※

$(T,S) \models E(\neg G \; \mathcal{U} \; (\neg F \wedge \neg G)) \vee E \Box \neg G$

iff $(T,S) \models E(\neg G \; \mathcal{U} \; (\neg F \wedge \neg G))$ or $(T,S) \models E \Box \neg G$;

(**) iff there exists a computation $\pi = S_0 \to S_1 \cdots S_n^{\pm}$ with $S_0 = S$
such that $\forall m \geq 0 \; (T, S_m) \models \neg G$.

or

there exists a computation $\pi = S_0 \to S_1 \to \cdots$ with $S_0 = S$
such that $\exists n_0 \geq 0$ such that $(T, S_{n_0}) \models \neg F \wedge \neg G$.
and $\forall k < n_0 : (T, S_k) \models \neg G$.

we now show that

(*) iff (**).

(*) $\Rightarrow$ (**) | Proof If there exists a computation $\pi = S_0 \to S_1 \to \cdots$ with $S_0 = S$
such that for all $m \geq 0$, $(T, S_m) \models \neg G$ then (**) holds.

Assume now that there exists a computation $\pi = S_0 \to S_1 \to \cdots$ with $S_0 = S$
such that $\exists m_0$ with $(T, m_0) \models G$, $(T, m) \models \neg G$ for all $m < m_0$
and $\exists k \leq m_0 - 1$ such that $(T, S_k) \models \neg F$.

We choose n_0 to be the $k \leq m_0 - 1$ for which $(T, S_k) \models \neg F$.
Then $(T, S_k) \models \neg G$ because $k < m_0$, so $(T, n_0) \models \neg F \wedge \neg G$.
In addition, we know that for all $k \leq n_0 \leq m_0 - 1$
we have $(T, S_k) \models \neg G$. Thus (**) holds also in this case.

(**) $\Rightarrow$ (*). Proof If there exists a computation $\pi = S_0 \to S_1 \to \cdots$ with $S_0 = S$
such that $\forall m \geq 0$, $(T, S_m) \models \neg G$, then (*) holds.

this is not the case and

Assume now that there exists a computation
$\pi = S_0 \to S_1 \to \cdots$ with $S_0 = S$ such that
$\exists n_0 \geq 0$ such that $(T, S_{n_0}) \models \neg F \wedge \neg G$ and
$\forall k \leq n_0 : (T, S_k) \models \neg G$.
Then : $\exists m_0 \geq 0$ such that $(T, S_{m_0}) \models G$, and
$\forall m < m_0 : (T, S_m) \models \neg G$.
$\exists k = n_0 < m_0$ such that $(T, S_k) \models \neg F$.
Thus (*) holds also in this case.