Formal Specification and Verification

Classical logic (5)

13.11.2018

Viorica Sofronie-Stokkermans

e-mail: sofronie@uni-koblenz.de

Until now

- Propositional logic
- First order logic

  Syntax:

  (Many sorted) Signature

  Terms

  Atomic formulae

  Formulae

Example: Specifying LISP lists

Signature:

$\Sigma_{\text{Lists}} = (\Omega_{\text{Lists}}, \Pi_{\text{Lists}})$

$\Omega_{\text{Lists}} = \{\text{car}/1, \text{cdr}/1, \text{cons}/2\}$

$\Pi_{\text{Lists}} = \emptyset$

Examples of formulae:

$\forall x, y \quad \text{car}(\text{cons}(x, y)) \approx x$

$\forall x, y \quad \text{cdr}(\text{cons}(x, y)) \approx y$

$\forall x \quad \text{cons}(\text{car}(x), \text{cdr}(x)) \approx x$

Many-sorted signatures

Example:

Signature

$S = \{\text{array}, \text{index}, \text{element}\}$ set of sorts

$\Omega = \{\text{read}, \text{write}\}$

$$a(\text{read}) = \text{array} \times \text{index} \to \text{element}$$
$$a(\text{write}) = \text{array} \times \text{index} \times \text{element} \to \text{array}$$

$\Pi = \emptyset$

$X = \{X_s \mid s \in S\}$

Examples of formulae:

$$\forall x : \text{array} \;\; \forall i : \text{index} \;\; \forall j : \text{index} \;\; (i \approx j \to \text{write}(x, i, \text{read}(x, j)) \approx x)$$

$$\forall x : \text{array} \;\; \forall y : \text{array} \;\; (x \approx y \leftrightarrow \forall i : \text{index} \;\; (\text{read}(x, i) \approx \text{read}(y, i)))$$

Bound and Free Variables

In QxF, $Q \in \{\exists, \forall\}$, we call F the scope of the quantifier Qx.
An *occurrence* of a variable x is called bound, if it is inside the scope of a quantifier Qx.
Any other occurrence of a variable is called free.

Formulas without free variables are also called closed formulas or sentential forms.

Formulas without variables are called ground.

Bound and Free Variables

Example:

$$\overbrace{\forall\, y \quad (\forall\, \overbrace{x \quad p(x)}^{\textit{scope}} \quad \rightarrow \quad q(x, y))}^{\textit{scope}}$$

The occurrence of y is bound, as is the first occurrence of x. The second occurrence of x is a free occurrence.

Substitutions

Substitution is a fundamental operation on terms and formulas that occurs in all inference systems for first-order logic.

In general, substitutions are mappings

$$\sigma : X \to \mathsf{T}_\Sigma(X)$$

such that the domain of σ, that is, the set

$$\mathit{dom}(\sigma) = \{x \in X \mid \sigma(x) \neq x\},$$

is finite. The set of variables introduced by σ, that is, the set of variables occurring in one of the terms $\sigma(x)$, with $x \in \mathit{dom}(\sigma)$, is denoted by $\mathit{codom}(\sigma)$.

Substitutions

Substitutions are often written as $[s_1/x_1, \ldots, s_n/x_n]$, with x_i pairwise distinct, and then denote the mapping

$$[s_1/x_1, \ldots, s_n/x_n](y) = \begin{cases} s_i, & \text{if } y = x_i \\ y, & \text{otherwise} \end{cases}$$

We also write $x\sigma$ for $\sigma(x)$.

The modification of a substitution σ at x is defined as follows:

$$\sigma[x \mapsto t](y) = \begin{cases} t, & \text{if } y = x \\ \sigma(y), & \text{otherwise} \end{cases}$$

Why Substitution is Complicated

We define the application of a substitution σ to a term t or formula F by structural induction over the syntactic structure of t or F by the equations depicted on the next page.

In the presence of quantification it is surprisingly complex:
We need to make sure that the (free) variables in the codomain of σ are not *captured* upon placing them into the scope of a quantifier Qy, hence the bound variable must be renamed into a "fresh", that is, previously unused, variable z.

Application of a Substitution

"Homomorphic" extension of σ to terms and formulas:

$$
\begin{aligned}
f(s_1, \ldots, s_n)\sigma &= f(s_1\sigma, \ldots, s_n\sigma) \\
\bot\sigma &= \bot \\
\top\sigma &= \top \\
p(s_1, \ldots, s_n)\sigma &= p(s_1\sigma, \ldots, s_n\sigma) \\
(u \approx v)\sigma &= (u\sigma \approx v\sigma) \\
\neg F\sigma &= \neg(F\sigma) \\
(F \rho G)\sigma &= (F\sigma \,\rho\, G\sigma)\ ; \quad \text{for each binary connective } \rho \\
(Qx\, F)\sigma &= Qz\,(F\,\sigma[x \mapsto z])\ ; \quad \text{with } z \text{ a fresh variable}
\end{aligned}
$$

2.2 Semantics

To give semantics to a logical system means to define a notion of truth for the formulas. The concept of truth that we will now define for first-order logic goes back to Tarski.

As in the propositional case, we use a two-valued logic with truth values "true" and "false" denoted by 1 and 0, respectively.

Structures

A Σ-algebra (also called Σ-interpretation or Σ-structure) is a triple

$$\mathcal{A} = (U,\ (f_{\mathcal{A}} : U^n \to U)_{f/n \in \Omega},\ (p_{\mathcal{A}} \subseteq U^m)_{p/m \in \Pi})$$

where $U \neq \emptyset$ is a set, called the universe of $\mathcal{A}$.

Normally, by abuse of notation, we will have $\mathcal{A}$ denote both the algebra and its universe.

By $\Sigma - \mathrm{Alg}$ we denote the class of all Σ-algebras.

Many-sorted Structures

A many-sorted Σ-algebra (also called Σ-interpretation or Σ-structure), where $\Sigma = (S, \Omega, \Pi)$ is a triple

$$\mathcal{A} = (\{U_s\}_{s \in S},\ (f_{\mathcal{A}} : U_{s_1} \times \ldots \times U_{s_n} \to U_s)_{\substack{f \in \Omega, \\ a(f) = s_1 \ldots s_n \to s}}\ \ (p_{\mathcal{A}} : U_{s_1} \times \ldots \times U_{s_m} \to \{0,1\})_{\substack{p \in \Pi \\ a(p) = s_1 \ldots s_m}}\ )$$

where $U \neq \emptyset$ is a set, called the universe of $\mathcal{A}$.

Assignments

A variable has no intrinsic meaning. The meaning of a variable has to be defined externally (explicitly or implicitly in a given context) by an assignment.

A (variable) assignment, also called a valuation (over a given Σ-algebra $\mathcal{A}$), is a map $\beta : X \to \mathcal{A}$.

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Many-sorted case:

$\beta = \{\beta_s\}_{s \in S}, \beta_s : X_s \to U_s$

Value of a Term in $\mathcal{A}$ with Respect to β

By structural induction we define

$$\mathcal{A}(\beta) : \mathsf{T}_\Sigma(X) \to \mathcal{A}$$

as follows:

$$\mathcal{A}(\beta)(x) = \beta(x), \qquad x \in X$$

$$\mathcal{A}(\beta)(f(s_1, \ldots, s_n)) = f_{\mathcal{A}}(\mathcal{A}(\beta)(s_1), \ldots, \mathcal{A}(\beta)(s_n)), \qquad f/n \in \Omega$$

Value of a Term in $\mathcal{A}$ with Respect to β

In the scope of a quantifier we need to evaluate terms with respect to modified assignments. To that end, let $\beta[x \mapsto a] : X \to \mathcal{A}$, for $x \in X$ and $a \in \mathcal{A}$, denote the assignment

$$\beta[x \mapsto a](y) := \begin{cases} a & \text{if } x = y \\ \beta(y) & \text{otherwise} \end{cases}$$

Truth Value of a Formula in $\mathcal{A}$ with Respect to β

$\mathcal{A}(\beta) : \mathsf{F}_\Sigma(X) \to \{0,1\}$ is defined inductively as follows:

$$
\begin{aligned}
\mathcal{A}(\beta)(\bot) &= 0 \\
\mathcal{A}(\beta)(\top) &= 1 \\
\mathcal{A}(\beta)(p(s_1,\ldots,s_n)) &= p_{\mathcal{A}}(\mathcal{A}(\beta)(s_1),\ldots,\mathcal{A}(\beta)(s_n)) \\
\mathcal{A}(\beta)(s \approx t) &= 1 \quad \Leftrightarrow \quad \mathcal{A}(\beta)(s) = \mathcal{A}(\beta)(t) \\
\mathcal{A}(\beta)(\neg F) &= 1 \quad \Leftrightarrow \quad \mathcal{A}(\beta)(F) = 0 \\
\mathcal{A}(\beta)(F \rho G) &= \mathsf{B}_\rho(\mathcal{A}(\beta)(F), \mathcal{A}(\beta)(G)) \\
&\qquad \text{with } \mathsf{B}_\rho \text{ the Boolean function associated with } \rho \\
\mathcal{A}(\beta)(\forall x F) &= \min_{a \in U}\{\mathcal{A}(\beta[x \mapsto a])(F)\} \\
\mathcal{A}(\beta)(\exists x F) &= \max_{a \in U}\{\mathcal{A}(\beta[x \mapsto a])(F)\}
\end{aligned}
$$

Example

The "Standard" Interpretation for Peano Arithmetic:

$$
\begin{array}{rcl}
U_{\mathbb{N}} & = & \{0, 1, 2, \ldots\} \\
0_{\mathbb{N}} & = & 0 \\
s_{\mathbb{N}} : U_{\mathbb{N}} \to U_{\mathbb{N}} & & s_{\mathbb{N}}(n) = n + 1 \\
+_{\mathbb{N}} : U_{\mathbb{N}}^2 \to U_{\mathbb{N}} & & +_{\mathbb{N}}(n, m) = n + m \\
*_{\mathbb{N}} : U_{\mathbb{N}}^2 \to U_{\mathbb{N}} & & *_{\mathbb{N}}(n, m) = n * m \\
\leq_{\mathbb{N}} : U_{\mathbb{N}}^2 \to \{0, 1\} & & \leq_{\mathbb{N}} (n, m) = 1 \text{ iff } n \text{ less than or equal to } m \\
<_{\mathbb{N}} : U_{\mathbb{N}}^2 \to \{0, 1\} & & \leq_{\mathbb{N}} (n, m) = 1 \text{ iff } n \text{ less than } m
\end{array}
$$

Note that $\mathbb{N}$ is just one out of many possible Σ_{PA}-interpretations.

Example

Values over $\mathbb{N}$ for Sample Terms and Formulas:

Under the assignment $\beta : x \mapsto 1, y \mapsto 3$ we obtain

$$\begin{aligned}
\mathbb{N}(\beta)(s(x) + s(0)) &= 3 \\
\mathbb{N}(\beta)(x + y \approx s(y)) &= 1 \\
\mathbb{N}(\beta)(\forall x, y(x + y \approx y + x)) &= 1 \\
\mathbb{N}(\beta)(\forall z\ z \leq y) &= 0 \\
\mathbb{N}(\beta)(\forall x \exists y\ x < y) &= 1
\end{aligned}$$

2.3 Models, Validity, and Satisfiability

F is valid in $\mathcal{A}$ under assignment β:

$$\mathcal{A}, \beta \models F \quad :\Leftrightarrow \quad \mathcal{A}(\beta)(F) = 1$$

F is valid in $\mathcal{A}$ ($\mathcal{A}$ is a model of F):

$$\mathcal{A} \models F \quad :\Leftrightarrow \quad \mathcal{A}, \beta \models F, \text{ for all } \beta \in X \to U_{\mathcal{A}}$$

F is valid (or is a tautology):

$$\models F \quad :\Leftrightarrow \quad \mathcal{A} \models F, \text{ for all } \mathcal{A} \in \Sigma\text{-alg}$$

F is called satisfiable iff there exist $\mathcal{A}$ and β such that $\mathcal{A}, \beta \models F$.
Otherwise F is called unsatisfiable.

Entailment and Equivalence

F entails (implies) G (or G is a consequence of F), written $F \models G$

$:\Leftrightarrow$ for all $\mathcal{A} \in \Sigma$-alg and $\beta \in X \to U_{\mathcal{A}}$,
whenever $\mathcal{A}, \beta \models F$ then $\mathcal{A}, \beta \models G$.

F and G are called equivalent

$:\Leftrightarrow$ for all $\mathcal{A} \in \Sigma$-alg und $\beta \in X \to U_{\mathcal{A}}$ we have
$\mathcal{A}, \beta \models F \quad \Leftrightarrow \quad \mathcal{A}, \beta \models G$.

Entailment and Equivalence

Proposition 2.6:

F entails G iff $(F \to G)$ is valid

Proposition 2.7:

F and G are equivalent iff $(F \leftrightarrow G)$ is valid.

Extension to sets of formulas N in the "natural way", e.g., $N \models F$

$:\Leftrightarrow$ for all $\mathcal{A} \in \Sigma$-alg and $\beta \in X \to U_{\mathcal{A}}$:
if $\mathcal{A}, \beta \models G$, for all $G \in N$, then $\mathcal{A}, \beta \models F$.

Validity vs. Unsatisfiability

Validity and unsatisfiability are just two sides of the same medal as explained by the following proposition.

Proposition 2.8:

$$F \text{ valid} \quad \Leftrightarrow \quad \neg F \text{ unsatisfiable}$$

Hence in order to design a theorem prover (validity checker) it is sufficient to design a checker for unsatisfiability.

Q: In a similar way, entailment $N \models F$ can be reduced to unsatisfiability. How?

Algorithmic Problems

Validity(F): $\models F$?

Satisfiability(F): F satisfiable?

Entailment(F,G): does F entail G?

Model($\mathcal{A}$,F): $\mathcal{A} \models F$?

Solve($\mathcal{A}$,F): find an assignment β such that $\mathcal{A}, \beta \models F$

Solve(F): find a substitution σ such that $\models F\sigma$

Abduce(F): find G with "certain properties" such that G entails F

Decidability/Undecidability

In 1931, Gödel published his incompleteness theorems in

"Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme"

(in English "On Formally Undecidable Propositions of Principia Mathematica and Related Systems").

He proved for any computable axiomatic system that is powerful enough to describe the arithmetic of the natural numbers (e.g. the Peano axioms or Zermelo-Fraenkel set theory with the axiom of choice), that:

- If the system is consistent, it cannot be complete.
- The consistency of the axioms cannot be proven within the system.

Decidability/Undecidability

These theorems ended a half-century of attempts, beginning with the work of Frege and culminating in Principia Mathematica and Hilbert's formalism, to find a set of axioms sufficient for all mathematics.

The incompleteness theorems also imply that not all mathematical questions are computable.

Consequences of Gödel's Famous Theorems

1. For most signatures Σ, validity is undecidable for Σ-formulas.
   (One can easily encode Turing machines in most signatures.)
2. For each signature Σ, the set of valid Σ-formulas is recursively enumerable.
   (This is proved by giving complete deduction systems.)
3. For $\Sigma = \Sigma_{PA}$ and $\mathbb{N}_* = (\mathbb{N}, 0, s, +, *)$, the theory $Th(\mathbb{N}_*)$ is not recursively enumerable.

These undecidability results motivate the study of subclasses of formulas (fragments) of first-order logic

Q: Can you think of any fragments of first-order logic for which validity is decidable?

Some Decidable Fragments/Problems

Validity/Satisfiability/Entailment: Some decidable fragments:

- Variable-free formulas without equality:
  satisfiability is NP-complete. (why?)
- Variable-free Horn clauses (clauses with at most one positive atom): entailment is decidable in linear time.
- Monadic class: no function symbols, all predicates unary;
  validity is NEXPTIME-complete.
- Q: Other decidable fragments of FOL (with variables)?
  Which methods for proving decidability?

Decidable problems.

Finite model checking is decidable in time polynomial in the size of the structure and the formula.

Calculi

There exist Hilbert style calculi and sequent calculi for first-order logic.

Checking satisfiability of formulae:

- Resolution
- Semantic tableaux

Verification: Logical theories

Theory of a Structure

Let $\mathcal{A} \in \Sigma$-alg. The (first-order) theory of $\mathcal{A}$ is defined as

$$Th(\mathcal{A}) = \{G \in \mathsf{F}_\Sigma(X) \mid \mathcal{A} \models G\}$$

Problem of axiomatizability:

For which structures $\mathcal{A}$ can one axiomatize $Th(\mathcal{A})$, that is, can one write down a formula F (or a recursively enumerable set F of formulas) such that

$$Th(\mathcal{A}) = \{G \mid F \models G\}?$$

Analogously for sets of structures.

Two Interesting Theories

Let $\Sigma_{Pres} = (\{0/0, s/1, +/2\}, \emptyset)$ and $\mathbb{Z}_+ = (\mathbb{Z}, 0, s, +)$ its standard interpretation on the integers.
$Th(\mathbb{Z}_+)$ is called Presburger arithmetic (M. Presburger, 1929).
(There is no essential difference when one, instead of $\mathbb{Z}$, considers the natural numbers $\mathbb{N}$ as standard interpretation.)

Presburger arithmetic is decidable in 3EXPTIME (D. Oppen, JCSS, 16(3):323–332, 1978), and in 2EXPSPACE, using automata-theoretic methods (and there is a constant $c \geq 0$ such that $Th(\mathbb{Z}_+) \notin \text{NTIME}(2^{2^{cn}})$).

Two Interesting Theories

However, $\mathbb{N}_* = (\mathbb{N}, 0, s, +, *)$, the standard interpretation of $\Sigma_{PA} = (\{0/0, s/1, +/2, */2\}, \emptyset)$, has as theory the so-called Peano arithmetic which is undecidable, not even recursively enumerable.

Note: The choice of signature can make a big difference with regard to the computational complexity of theories.

Logical theories

Syntactic view

first-order theory: given by a set $\mathcal{F}$ of (closed) first-order Σ-formulae.

the models of $\mathcal{F}$: $\quad \mathsf{Mod}(\mathcal{F}) = \{\mathcal{A} \in \Sigma\text{-alg} \mid \mathcal{A} \models G, \text{ for all } G \text{ in } \mathcal{F}\}$

Semantic view

given a class $\mathcal{M}$ of Σ-algebras

the first-order theory of $\mathcal{M}$: $\mathsf{Th}(\mathcal{M}) = \{G \in F_\Sigma(X) \text{ closed} \mid \mathcal{M} \models G\}$

Theories

$\mathcal{F}$ set of (closed) first-order formulae

$$\mathsf{Mod}(\mathcal{F}) = \{A \in \Sigma\text{-alg} \mid \mathcal{A} \models G, \text{ for all } G \text{ in } \mathcal{F}\}$$

$\mathcal{M}$ class of Σ-algebras

$$\mathsf{Th}(\mathcal{M}) = \{G \in F_\Sigma(X) \text{ closed} \mid \mathcal{M} \models G\}$$

$\mathsf{Th}(\mathsf{Mod}(\mathcal{F}))$ the set of formulae true in all models of $\mathcal{F}$

represents exactly the set of consequences of $\mathcal{F}$

Theories

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$\mathrm{Th}(\mathrm{Mod}(\mathcal{F}))$ the set of formulae true in all models of $\mathcal{F}$
represents exactly the set of consequences of $\mathcal{F}$

Note: $\mathcal{F} \subseteq \mathrm{Th}(\mathrm{Mod}(\mathcal{F}))$ (typically strict)

$\mathcal{M} \subseteq \mathrm{Mod}(\mathrm{Th}(\mathcal{M}))$ (typically strict)

Examples

1. Groups

Let $\Sigma = (\{e/0, */2, i/1\}, \emptyset)$

Let $\mathcal{F}$ consist of all (universally quantified) group axioms:

$$\begin{array}{lll} \forall x, y, z & x * (y * z) \approx (x * y) * z & \\ \forall x & x * i(x) \approx e & \wedge \quad i(x) * x \approx e \\ \forall x & x * e \approx x & \wedge \quad e * x \approx x \end{array}$$

Every group $\mathcal{G} = (G, e_G, *_G, i_G)$ is a model of $\mathcal{F}$

$\mathsf{Mod}(\mathcal{F})$ is the class of all groups

$\mathcal{F} \subset \mathsf{Th}(\mathsf{Mod}(\mathcal{F}))$

Examples

2. Linear (positive)integer arithmetic

Let $\Sigma = (\{0/0, s/1, +/2\}, \{\leq /2\})$

Let $\mathbb{Z}_+ = (\mathbb{Z}, 0, s, +, \leq)$ the standard interpretation of integers.

$\{\mathbb{Z}_+\} \subset \mathsf{Mod}(\mathsf{Th}(\mathbb{Z}_+))$

3. Uninterpreted function symbols

Let $\Sigma = (\Omega, \Pi)$ be arbitrary

Let $\mathcal{M} = \Sigma$-alg be the class of all Σ-structures

The theory of uninterpreted function symbols is Th(Σ-alg) the family of all first-order formulae which are true in all Σ-algebras.

Examples

4. Lists

Let $\Sigma = (\{\mathsf{car}/1, \mathsf{cdr}/1, \mathsf{cons}/2\}, \emptyset)$

Let $\mathcal{F}$ be the following set of list axioms:

$$\begin{aligned} \mathsf{car}(\mathsf{cons}(x, y)) &\approx x \\ \mathsf{cdr}(\mathsf{cons}(x, y)) &\approx y \\ \mathsf{cons}(\mathsf{car}(x), \mathsf{cdr}(x)) &\approx x \end{aligned}$$

$\mathsf{Mod}(\mathcal{F})$ class of all models of $\mathcal{F}$

$\mathsf{Th}_{\mathsf{Lists}} = \mathsf{Th}(\mathsf{Mod}(\mathcal{F}))$ theory of lists (axiomatized by $\mathcal{F}$)

"Most general" models

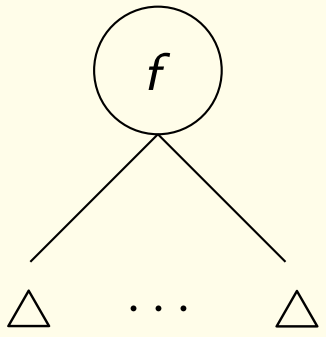
We assume that $\Pi = \emptyset$.

Term algebras

A term algebra (over Σ) is a Σ-algebra $\mathcal{A}$ such that

- $U_{\mathcal{A}} = \mathsf{T}_\Sigma$ (= the set of ground terms over Σ)
- $f_{\mathcal{A}} : (s_1, \ldots, s_n) \mapsto f(s_1, \ldots, s_n), \;\; f/n \in \Omega$

$$f_{\mathcal{A}}(\triangle, \ldots, \triangle) =$$



Term algebras

In other words, *values are fixed* to be ground terms and *functions are fixed* to be the term constructors.