Formal Specification and Verification

Temporal logic (2)

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Linear Time Logic

Semantics

- Transition systems $(S, \rightarrow, L)$
  (with the property that for every $s \in S$ there exists $s' \in S$ with $s \rightarrow s'$ i.e. no state of the system can "deadlock"[a])

  Transition systems are also simply called models in what follows.

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[a]This is a technical convenience, and in fact it does not represent any real restriction on the systems we can model. If a system did deadlock, we could always add an extra state s_d representing deadlock, together with new transitions $s \rightarrow s_d$ for each s which was a deadlock in the old system, as well as $s_d \rightarrow s_d$.

Linear Time Logic

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  (with the property that for every $s \in S$ there exists $s' \in S$ with $s \to s'$ i.e. no state of the system can "deadlock"[a])

  Transition systems are also simply called models in what follows.

- Computation (execution, path) in a model $(S, \to, L)$
  infinite sequence of states $\pi = s_0, s_1, s_2, \ldots$ in S such that for each $i \geq 0$, $s_i \to s_{i+1}$.
  We write the path as $s_0 \to s_1 \to s_2 \to \ldots$.

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Linear Time Logic

Semantics

Let $TS = (S, \rightarrow, L)$ be a model and $\pi = s_0 \rightarrow$ ... be a path in TS.

Whether π satisfies an LTL formula is defined by the satisfaction relation $\models$ as follows:

- $\pi \models \top$
- $\pi \not\models \bot$
- $\pi \models p$ iff $p \in L(s_0)$, if $p \in \Pi$
- $\pi \models \neg F$ iff $\pi \not\models F$
- $\pi \models F \wedge G$ iff $\pi \models F$ and $\pi \models G$
- $\pi \models F \vee G$ iff $\pi \models F$ or $\pi \models G$
- $\pi \models \bigcirc F$ iff $\pi^1 \models F$
- $\pi \models F \mathcal{U} G$ iff $\exists m \geq 0$ $s.t.$ $\pi^m \models G$ and $\forall k \in \{0, \ldots, m-1\} : \pi^k \models F$

Linear Time Logic

Alternative way of defining the semantics:

An LTL structure M is an infinite sequence $S_0 S_1 \ldots$ with $S_i \subseteq \Pi$ for all $i \geq 0$. We define satisfaction of LTL formulas in M at time points $n \in \mathbb{N}$ as follows:

- $M, n \models p$ iff $p \in S_n, \qquad$ if $p \in \Pi$
- $M, n \models F \wedge G$ iff $M, n \models F$ and $M, n \models G$
- $M, n \models F \vee G$ iff $M, n \models F$ or $M, n \models G$
- $M, n \models \neg F$ iff $M, n \not\models F$
- $M, n \models \bigcirc F$ iff $M, n+1 \models F$
- $M, n \models F \mathcal{U} G$ iff $\exists m \geq n \; s.t. \; M, m \models G$ and $\forall k \in \{n, \ldots, m-1\} : M, k \models F$

Note that the time flow $(\mathbb{N}, <)$ is implicit.

Abbreviations

- The future diamond

  $\Diamond\phi := \top\mathcal{U}\phi$

  $\pi \models \Diamond\phi$ iff $\exists m \geq 0 : \pi^m \models \phi$

- The future box

  $\Box\phi := \neg\Diamond\neg\phi$

  $\pi \models \Box\phi$ iff $\forall m \geq 0 : \pi^m \models \phi$

- The infinitely often operator

  $\Diamond^\infty\phi := \Box\Diamond\phi$

  $\pi \models \Diamond^\infty\phi$ iff $\{m \geq 0 \mid \pi^m \models \phi\}$ is infinite

- The almost everywhere operator

  $\Box^\infty\phi := \Diamond\Box\phi$

  $\pi \models \Box^\infty\phi$ iff $\{m \geq 0 \mid \pi^m \not\models \phi\}$ is finite.

Abbreviations

- The release operator
  $\phi\mathcal{R}\psi := \neg(\neg\phi\mathcal{U}\neg\psi)$

  $\pi \models \phi\mathcal{R}\psi$ iff $(\exists m \geq 0 : \pi^m \models \phi$ and $\forall k < m$: $\pi^k \models \psi)$ or
  $(\forall k \geq 0 : \pi^k \models \psi)$

  Read as
  "ψ always holds unless released by ϕ" i.e.,
  "ψ holds permanently up to and including the first point where ϕ holds (such an ϕ-point need not exist at all)".

- The strict until operator:
  $F\mathcal{U}^{<}G := \bigcirc(F\mathcal{U}G)$
  $\pi \models F\mathcal{U}^{<}G$ iff $\exists m > 0 : \pi^m \models G \wedge \forall k \in \{1, 2, \ldots, m-1\}, \pi^k \models F$

Temporal Properties

A temporal property is a set of LTL structures
(those on which the property is true).

Thus, a temporal property P can be defined using an LTL formula F:

$$P = \{M \mid M, 0 \models F\}.$$

When given a transition system TS representing a reactive system and an LTL formula F representing a temporal property,

TS satisfies F if $M, 0 \models F$ for all traces M of TS.

In this case, we write $TS \models F$.

Typical properties of reactive systems that need to be checked during verification are safety properties, liveness properties, and fairness properties.

Safety properties

Intuitively, a safety property asserts that "nothing bad happens"

general form: Condition $\rightarrow \Box F_{\mathsf{Safe}}$

Examples of safety properties:

- **Mutual Exclusion.** For the example:

$$\Box(\neg((A = 2) \wedge (B = 2)))$$

- **Freedom from Deadlocks:** At any time, some process should be enabled:

$$\Box(enabled_1 \vee \cdots \vee enabled_k)$$

- **Partial Correctness:** If F is satisfied when the program starts, then G will be satisfied if the program reaches a distinguished state:

$$F \rightarrow \Box(\mathsf{Dist} \rightarrow G)$$

where Dist $\in \Pi$ marks the distinguished state.

Liveness properties

Intuitively, a liveness property asserts that "something good will happen"

Examples of liveness properties:

- **Guaranteed Accessibility.** For the example:

$$\Box(A = 1 \rightarrow \Diamond(A = 2)) \land \Box(B = 1 \rightarrow \Diamond(B = 2))$$

- **Responsiveness:** If a request is issued, it will eventually be granted:

$$\Box(\mathsf{req} \rightarrow \Diamond \mathsf{grant})$$

- **Total Correctness:** If F is satisfied when the program starts, then the program terminates in a distinguished state where G is satisfied:

$$\phi \rightarrow \Diamond(\mathsf{Dist} \land G)$$

Note that, in contrast, partial correctness is a safety property.

Fairness properties

When modelling concurrent systems, it is usually important to make some fairness assumptions. Assume that there are k processes, that $\mathsf{enabled}_i \in \Pi$ is true in a state s if process $\#i$ is enabled in s for execution, and that $\mathsf{executed}_i$ is true in a state s if process $\#i$ has been executed to reach s.

Examples of fairness properties

- **Unconditional Fairness:** Every process is executed infinitely often:

$$\bigwedge_{1 \leq i \leq k} \Diamond^{\infty} \mathsf{executed}_i$$

  Unconditional fairness is appropriate when processes can (and should!) be executed and any time. This is not always the case.

Fairness properties

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Examples of fairness properties

- **Strong Fairness:** Every process enabled infinitely often is executed infinitely often:

$$\bigwedge_{1 \leq i \leq k} (\Diamond^{\infty}\mathsf{enabled}_i \rightarrow \Diamond^{\infty}\mathsf{executed}_i)$$

  Processes enabled only finitely often need not be guaranteed to be executed: they eventually and forever retract being enabled.

Fairness properties

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Examples of fairness properties

- **Weak Fairness:** Every process enabled almost everywhere is executed infinitely often.

$$\bigwedge_{1 \leq i \leq k} (\Box^{\infty}\mathsf{enabled}_i \rightarrow \Diamond^{\infty}\mathsf{executed}_i)$$

  This means that a process cannot be enabled constantly in an infinite interval without being executed in this interval.

Semantics, Overview

TS transition system, $\pi = s_0 \to s_1 \to \ldots$ path in TS.

$\pi \models F$ iff $L(s_0) \ldots L(s_n), 0 \models F$

s state of TS.

$s \models F$ iff $(\forall \pi$ path starting in $s : \pi \models F)$

$TS \models F$	iff	$\pi \models F$ for all paths π
	iff	$s \models F$ for all states s of TS
	iff	$M, 0 \models F$ for all traces M of TS

Satisfiability

An LTL formula F is satisfiable
iff there exists a transition system TS and a path π such that $\pi \models F$
iff there exists a LTL structures M and $n \geq 0$ such that $M, n \models F$

Such a TS/structure is called a model of F.

In verification, satisfiability can be used to detect contradictory properties, i.e., properties that are satisfied by no computation of any reactive system.

Example: The following property is contradictory (unsatisfiable):

$$p \land \Box(p \to \bigcirc p) \land \Diamond \neg p$$

Satisfiability

When using LTL for verification, we are usually interested in whether a formula holds at point 0 of an LTL structure.

Lemma. Every satisfiable LTL formula F has a model M with $M, 0 \models F$.

Proof (Sketch)

Let $M, n \models F$, and let M' be the model obtained from M by dropping all time points $0, \ldots, n-1$. Thus, time point n in M is time point 0 in M'.

It is easy to prove by induction on the structure of G that, for all LTL formulas G and $i \geq 0$, we have $M', i \models G$ iff $M, n+i \models G$.

It follows that $M', 0 \models F$.

Semantics: Variants

Sometimes in the literature the models are of the form:

$TS = (S, \rightarrow, S_i, L)$, where S_i is a set of initial states.

Then:

$TS \models F$ iff $\pi \models F$ for all initial paths π

iff $s \models F$ for all initial states s of TS

Satisfiability

LTL satisfiability can be decided using automata on infinite words (Büchi automata).

Model checking

The LTL model checking problem is as follows: given a transition system $TS = (S, \rightarrow, S_i, L)$ and an LTL formula F, check whether $TS \models F$.

Model checking

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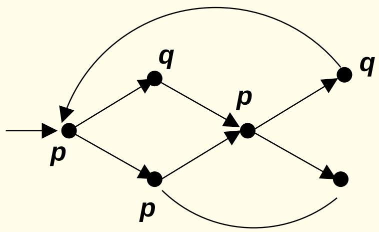
Recall: this is the case iff

- all initial paths π of TS satisfy $\pi \models F$, iff
- for all initial states s of TS we have: $s \models F$.

Example:

The following transition system satisfies $\Box(q \rightarrow \bigcirc \bigcirc \bigcirc p)$.

It does not satisfy $\Box(p \rightarrow p \mathcal{U} q)$.



Connection to First-Order Logic

Another characterization of temporal properties that can be expressed in LTL is obtained by relating LTL to the monadic first-order theory of the natural numbers.

Let $FO^{<}$ denote the following first-order language:

- no function symbols and constants;
- binary predicate symbols: "suc" for successor, an order predicate $<$, and equality;
- countably infinite supply of unary predicates.

Connection to First-Order Logic

We may interpret formulas of $FO^{<}$ on LTL structures:

- quantification is over $\mathbb{N}$,
- the binary predicates are interpreted in the obvious way, and
- the unary predicates are identified with propositional variables.

Connection to First-Order Logic

We write $\phi(x_1, ..., x_n)$ to indicate that the variables in the $FO^<$ formula ϕ are $x_1, ..., x_n$.

For an $FO^<$ formula $\phi(x_1, ..., x_n)$, an LTL structure M, and $n_1, ..., n_k \in \mathbb{N}$, we write $M \models \phi[n_1, ..., n_k]$ if ϕ is true in M with variable x_i bound to value n_i, for $1 \leq i \leq k$.

Examples:

- For $\phi(x_1, x_2) = \neg p(x_1) \wedge p(x_2) \wedge \forall x_3.(x_1 < x_3 \rightarrow \neg q(x_3))$, we have $\varnothing\{p\} \ldots \{p\} \ldots \models \phi[0, 1]$.
- The following formula $\phi(x)$ expresses that there exists a future point that agrees with the current point (identified by the free variable) on the unary predicates $p_1, ..., p_n$:

$$\phi(x) = \exists y(x < y \wedge \bigwedge_{1 \leq i \leq n} (p_i(x) \leftrightarrow p_i(y)))$$

Connection to First-Order Logic

We say that an $FO^{<}$ formula $\phi(x)$ with exactly one free variable is equivalent to an LTL formula F if for all LTL models M and $n \in \mathbb{N}$ we have

$$M, n \models F \quad \text{iff} \quad M \models \phi[n].$$

Theorem: For every LTL formula F, there exists an equivalent $FO^{<}$ formula.

Proof The following translation $\mu : F_{LTL} \to FO^{<}$ takes LTL formulas F to equivalent $FO^{<}$ formulae:

$\mu(\top) = \top$; $\mu(\bot) = \bot$; $\mu(p)(x) = p(x)$ for every propositional variable p

$\mu(\neg F)(x) = \neg\, \mu(F)(x)$

$\mu(F \wedge G)(x) = \mu(F)(x) \wedge \mu(G)(x)$

$\mu(\bigcirc F)(x) = \exists y(suc(x, y) \wedge \mu(F)(y))$

$\mu(F \mathcal{U} G)(x) = \exists y(x \leq y \wedge \mu(G)(y) \wedge \forall z(x \leq z < y \to \mu(F)(z)))$

In the last two cases, variables y and z are newly introduced for every translation step.

Connection to First-Order Logic

What about the converse?

In general, are there $FO^{<}$ formulas $\phi(x)$ for which there is no equivalent LTL formula?

Connection to First-Order Logic

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In general, are there $FO^<$ formulas $\phi(x)$ for which there is no equivalent LTL formula?

Obviously there are: the formula $\exists y(y < x)$ states that there exists a previous time point – which cannot be expressed using only the future operators of LTL.

When we want to compare $FO^<$ with LTL, we should extend the latter with past-time temporal operators $\bigcirc^-$ and $\mathcal{S}$.

$M, n \models \bigcirc^- F$ iff $n > 0$ and $M, n-1 \models F$

$M, n \models F \mathcal{S} G$ iff $\exists m \leq n : M, m \models G$ and $M, k \models F$ for all $k \in \{m+1, ..., n\}$

Connection to First-Order Logic

This variant of LTL is called LTL with past operators (LTLP).

Connection to First-Order Logic

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Theorem (Kamp) For every $FO^<$ formula with one free variable, there exists an equivalent LTLP formula.

Proof. Out of the scope of this lecture.

Branching Time Logic: CTL

When doing model checking, we effectively use LTL in a branching time environment:

> Every state in a transition system that has more than a single successor gives rise to a “branching” in time.

This is reflected by the fact that usually, a transition system has more than a single computation.

Branching time logics allow us to explicitly talk about such branches in time.

CTL: Syntax

The class of computational tree logic (CTL) formulas is the smallest set such that

- $\top, \bot$ and each propositional variable $P \in \Pi$ are formulae;
- if F, G are formulae, then so are $F \wedge G, F \vee G, \neg F$;
- if F, G are formulae, then so are
  $A \bigcirc F$ and $E \bigcirc F$,
  $A(F \mathcal{U} G)$ and $E(F \mathcal{U} G)$.

The symbols A and E are called path quantifiers.

Abbreviations

Apart from the Boolean abbreviations, we use:

$A\Diamond F$ for $A(\top \mathcal{U} F)$

$E\Diamond F$ for $E(\top \mathcal{U} F)$

$A\Box F$ for $\neg E\Diamond\neg F$

$E\Box F$ for $\neg A\Diamond\neg F$

Note that formulas such as $E(\Box q \wedge \Diamond p)$ are not CTL formulas.

CTL: Semantics

Let $T = (S, \rightarrow, L)$ be a transition system. We define satisfaction of CTL formulas in T at states $s \in S$ as follows:

$(T, s) \models p$	iff	$p \in L(s)$
$(T, s) \models \neg F$	iff	$(T, s) \models F$ is not the case
$(T, s) \models F \wedge G$	iff	$(T, s) \models F$ and $(T, s) \models G$
$(T, s) \models F \vee G$	iff	$(T, s) \models F$ or $(T, s) \models G$
$(T, s) \models E \bigcirc F$	iff	$(T, t) \models F$ for some $t \in S$ with $s \rightarrow t$
$(T, s) \models A \bigcirc F$	iff	$(T, t) \models F$ for all $t \in S$ with $s \rightarrow t$
$(T, s) \models A(F \mathcal{U} G)$	iff	for all computations $\pi = s_0 s_1 \ldots$ of T with $s_0 = s$, there is an $m \geq 0$ such that $(T, s_m) \models G$ and $(T, s_k) \models F$ for all $k < m$
$(T, s) \models E(F \mathcal{U} G)$	iff	there exists a computation $\pi = s_0 s_1 \ldots$ of T with $s_0 = s$, such that there is an $m \geq 0$ such that $(T, s_m) \models G$ and $(T, s_k) \models F$ for all $k < m$

Example of formulae in CTL

- $E\Diamond((A = 2) \wedge (B = 2))$
  It is possible to reach a state where both processes are in the critical section.
- $A\Box(\text{enabled}_1 \wedge \ldots \text{enabled}_k)$
  freedom from deadlocks (a safety property);
- $A\Box(\text{req} \rightarrow A\Diamond\text{grant})$
  every request will eventually be acknowledged (a liveness property);
- $A\Box(A\Diamond\text{enabled}_i)$
  process i is enabled infinitely often on every computation path (unconditional fairness)
- $A\Box(E\Diamond\text{Restart})$
  from every state it is possible to get to a restart state

Equivalence

We say that two CTL formulas F and G are (globally) equivalent
(written $F \equiv G$)
if, for all CTL structures $T = (S, \rightarrow, L)$ and $s \in S$, we have

$$T, s \models F \text{ iff } T, s \models G.$$

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Examples:

$\neg A \Diamond F \equiv E \Box \neg F$

$\neg E \Diamond F \equiv A \Box \neg F$

$\neg A \bigcirc F \equiv E \bigcirc \neg F$

$A \Diamond F \equiv A[\top \mathcal{U} F]$

$E \Diamond F \equiv E[\top \mathcal{U} F]$

CTL

Why is CTL called a tree logic?

Intuitively, it can talk about branching paths (which exists in a tree), but not about joining path (which do not exist in a tree).

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Let $T = (S, \rightarrow, L)$ be a transition system.

We define a tree-shaped transition systems $Tree(T) = (S', \rightarrow', L')$ as follows:

- S' is the set of all finite computations of T, i.e., $S' = \{s_0 \dots s_k \mid s_i \rightarrow s_{i+1} \text{ for all } i < k\}$;
- $\rightarrow' = \{(\pi, \pi') \in S' \times S' \mid \pi = qs, \pi' = \pi s' \text{ for some } s, s' \in S \text{ with } s \rightarrow s'\}$;
- $(P \in L'(\pi) \text{ iff } P \in L(s))$ if $\pi = s\pi'$ for some $\pi' \in \{\epsilon\} \cup S'$ and $s \in S$.

Tree(T) is called the unravelling of T. Observe that *Tree*(T) has no leaves because of the assumption that we have no deadlocks in T.

CTL

CTL formulas cannot distinguish between a state in a transition system and the corresponding states in the tree-shaped unravelling.

Lemma Let T be a transition system, s a state of T, $\pi = s_0 \dots s_k$ a state of $Tree(T)$ such that $s_k = s$, and F a CTL formula.

Then $(T, s) \models F$ iff $(Tree(T), \pi) \models F$.

Proof. By induction on the structure of F.