Universität Koblenz-Landau

FB 4 Informatik

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Exercises for "Non-Classical Logics"
Exercise sheet 6

Exercise 6.1: (*3 P*)
Use the method presented in the lecture for checking whether the formula

$$F := (P \Rightarrow Q) \vee (Q \Rightarrow P)$$

is a tautology in the Łukasiewicz logic $\mathcal{L}_{\aleph^1}$, with set of truth values $[0, 1]$. (For checking the satisfiability over $[0, 1]$ of the constraints you obtain use any method you like (e.g. make case distinctions).)

Exercise 6.2: (*3 P*)
Use the method presented in the lecture for showing that the formula

$$F := (P \vee \neg P)$$

is not a tautology in the Gödel logic with set of truth values $[0, 1]$. (For finding a satisfying assignment into $[0, 1]$ for the constraints you obtain – and thus a variable assignment $\mathcal{A}$ with $\mathcal{A}(F) \neq 1$ – you can use any method you like.)

Exercise 6.3: (*4 P*)
Consider the following operations on $[0, 1]$:

- $\sqcup : [0, 1]^2 \to [0, 1]$ defined by $x \sqcup y = (\neg x) \Rightarrow y$;
- $\sqcap : [0, 1]^2 \to [0, 1]$ defined by $x \sqcap y = \neg((\neg x) \sqcup (\neg y))$;

where $\neg, \Rightarrow$ are the Łukasiewicz negation resp. implication.

(1) Prove that:
  - $x \sqcup y = \mathsf{min}(1, x + y)$
  - $x \sqcap y = \mathsf{max}(0, x + y - 1)$

(2) Prove by induction that for every $m \in \mathbb{N}$, $m \geq 2$:

$$\sqcup^m x = \underbrace{x \sqcup x \sqcup \cdots \sqcup x}_{m \text{ times}} = \mathsf{min}(1, m \cdot x)$$

Infer from this that $\sqcap^m x = \underbrace{x \sqcap x \sqcap \cdots \sqcap x}_{m \text{ times}} = 1 - \mathsf{min}(1, m \cdot (1 - x))$.

Hint: Use the fact that $x \sqcap y = \neg((\neg x) \sqcup (\neg y))$.

(3) Use (2) to show that $\bigsqcup^{m-1} \neg x = 1$ if and only if $x \leq \frac{m-2}{m-1}$.

Supplementary exercises
(not obligatory; to be discussed in the next exercise session)

Exercise 6.4: *(4 P)*
With the notation in Exercise 6.3, show that:

(1) $(x \sqcup (\sqcap^{m-2} x)) = 1$ if and only if $x \geq \frac{m-2}{m-1}$.

(2) Let F be a propositional formula in the signature of Łukasiewicz logic, and let C_{m-1} be the following formula:

$$C_{m-1} := (\sqcup^{m-1} \neg P) \wedge (P \sqcup (\sqcap^{m-2} P))$$

where P is a propositional variable.

From (1) and 6.3(3) infer that for every valuation $\mathcal{A} : \Pi \to [0, 1]$ the following are equivalent:

(i) $\mathcal{A}(C_{m-1}) = 1$
(ii) $\mathcal{A}(P) = \frac{m-2}{m-1}$

Exercise 6.5: *(6 P)*
Let $\mathcal{L}_n, \mathcal{L}_m$ be the finitely-valued Łukasiewicz logics with truth values $W_n = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \dots, \frac{n-2}{n-1}, 1\}$ and respectively $W_m = \{0, \frac{1}{m-1}, \frac{2}{m-1}, \dots, \frac{m-2}{m-1}, 1\}$.

(1) Assume that $W_m \not\subseteq W_n$. Show that in this case $\frac{1}{m-1} \notin W_n$ and $\frac{m-2}{m-1} \notin W_n$.

(2) Let C_m be the formula defined in Exercise 6.4(2) and $G := \sqcup^{n-1} \neg C_{m-1}$. Show that:

(i) for every valuation $\mathcal{A} : \Pi \to W_m$ if $\mathcal{A}(P) = \frac{m-2}{m-1}$ then $\mathcal{A}(G) = 0$.
(ii) for every valuation $\mathcal{A} : \Pi \to W_n$, $\mathcal{A}(G) = 1$.

(3) Assume that $W_m \not\subseteq W_n$, hence, by (1), that $\frac{m-2}{m-1} \notin W_n$. Show that in this case G is an $\mathcal{L}_n$-tautology, but is not an $\mathcal{L}_m$-tautology.

(4) Assume that $\mathsf{Tautologies}(\mathcal{L}_n) \subseteq \mathsf{Tautologies}(\mathcal{L}_m)$.

(i) Show that then $W_m \subseteq W_n$.
(ii) Use the fact that $\frac{1}{m-1} \in W_n$ to show that $(n-1)$ is a multiple of $(m-1)$, i.e. there exists $k \in \mathbb{N}$ such that $n - 1 = k \cdot (m - 1)$.

Please submit your solution until Wednesday, December 7, 2011. Please do not forget to write your name on your solution.