Non-classical logics

Lecture 17: Description Logics

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Until now

Modal logic

Syntax

Semantics

- Kripke models
- global and local entailment; deduction theorem

Correspondence theory

First-order definability

Theorem proving in modal logics

Decidability

Today

Description logics

Description Logics

subfield of Knowledge Representation which is a subfield of AI.

- **Description**– comes from concept description (formal expression which determines a set of individuals with common properties)
- **Logics** – comes from the fact that the semantics of concept description can be defined using logic (a fragment of first-order logic)

Why description logics?

Examples of concepts

teaching assistant, undergraduate, professor

Examples of properties

Every teaching assistant is either not an undergraduated or a professor.

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Formal description in first-order logic

Unary predicates: $\texttt{Teaching-Assistant}, \texttt{Undergrad}, \texttt{Professor}$

$$\forall x \quad \texttt{Teaching-Assistant}(x) \rightarrow \neg\texttt{Undergrad}(x) \vee \texttt{Professor}(x)$$

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More concise description

Concept names: $\texttt{Teaching-Assistant}, \texttt{Undergrad}, \texttt{Professor}$

$\texttt{Teaching-Assistant} \ \sqsubseteq \ \neg\texttt{Undergrad} \ \sqcup \ \texttt{Professor}$

Why description logics?

If predicate logic is directly used without some kind of restriction, then

- the structure of the knowledge/information is lost;
- the expressive power is too high for having good computational properties and efficient procedures.

Example

$$\texttt{Teaching-Assistant} \sqsubseteq \neg\texttt{Undergrad} \sqcup \texttt{Professor}$$

$$\forall x \quad \texttt{Teaching-Assistant}(x) \rightarrow \neg\texttt{Undergrad}(x) \vee \texttt{Professor}(x)$$

A necessary condition in order to be a teaching assistant is to be either not undergraduated or a professor.

Example

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$\forall x \quad \texttt{Teaching-Assistant}(x) \rightarrow \neg\texttt{Undergrad}(x) \vee \texttt{Professor}(x)$

A necessary condition in order to be a teaching assistant is to be either not undergraduated or a professor.

When the left-hand side is an atomic concept, the "$\sqsubseteq$" symbol introduces a *primitive definition* – giving only necessary conditions.

$\texttt{Teaching-Assistant} \doteq \neg\texttt{Undergrad} \sqcup \texttt{Professor}$

$\forall x \quad \texttt{Teaching-Assistant}(x) \leftrightarrow \neg\texttt{Undergrad}(x) \vee \texttt{Professor}(x)$

The "$\doteq$" symbol introduces a real definition – with necessary and sufficient conditions.

In general, we can have complex concept expressions at the left-hand side as well.

The description logic ALC: Syntax

Concepts: • primitive concepts N_C

• complex concepts (built using constructors $\neg, \sqcap, \sqcup, \exists R, \forall R, \top, \bot$)

Roles: N_R

The description logic ALC: Syntax

Concepts: • primitive concepts N_C

• complex concepts (built using constructors $\neg, \sqcap, \sqcup, \exists R, \forall R, \top, \bot$)

Roles: N_R

Concepts:

$$
\begin{array}{lll}
C := & \top & \\
& |\bot & \\
& |A & \text{primitive concept} \\
& |C_1 \sqcap C_2 & \\
& |C_2 \sqcup C_2 & \\
& |\neg C & \\
& |\forall R.C & \\
& |\exists R.C &
\end{array}
$$

The description logic ALC: Semantics

Interpretations: $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$

- $C \in N_C \mapsto C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$
- $R \in N_R \mapsto R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$

We can also interpret "individuals" (as elements of $\Delta^{\mathcal{I}}$).

The description logic ALC

Syntax	Semantics	Name
A	$A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$	primitive concept
R	$R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$	primitive role
$\top$	$\Delta^{\mathcal{I}}$	top
$\bot$	$\emptyset$	bottom
$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$	complement
$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$	conjunction
$C \sqcup D$	$C^{\mathcal{I}} \cup D^{\mathcal{I}}$	disjunction
$\forall R.C$	$\{x \mid \forall y \;\; R^{\mathcal{I}}(x, y) \rightarrow y \in C^{\mathcal{I}}\}$	universal quantification (universal role restriction)
$\exists R.C$	$\{x \mid \exists y \;\; R^{\mathcal{I}}(x, y) \;\wedge y \in C^{\mathcal{I}}\}$	existential quantification (existential role restriction)

The description logic ALC: Semantics

- **Conjunction** is interpreted as *intersection* of sets of individuals.
- **Disjunction** is interpreted as *union* of sets of individuals.
- **Negation** is interpreted as *complement* of sets of individuals.

For every interpretation $\mathcal{I}$:

- $(\neg(C \sqcap D))^{\mathcal{I}} = (\neg C \sqcup \neg D)^{\mathcal{I}}$
- $(\neg(C \sqcup D))^{\mathcal{I}} = (\neg C \sqcap \neg D)^{\mathcal{I}}$
- $(\neg(\forall R.C))^{\mathcal{I}} = (\exists R.\neg C)^{\mathcal{I}}$
- $(\neg(\exists R.C))^{\mathcal{I}} = (\forall R.\neg C)^{\mathcal{I}}$

Knowledge Bases

- **Terminological Axioms (TBox):** $C \sqsubseteq D$, $C \doteq D$
  - $\texttt{Student} \doteq \texttt{Person} \sqcap \exists\texttt{NAME.String} \sqcap$
    $\exists\texttt{ADDRESS.String} \sqcap$
    $\exists\texttt{ENROLLED.Course}$
  - $\texttt{Student} \sqsubseteq \exists\texttt{ENROLLED.Course}$
  - $\exists\texttt{TEACHES.Course} \sqsubseteq \neg\texttt{Undergrad} \sqcup \texttt{Professor}$

- **Membership statements (ABox):** $C(a)$, $R(a, b)$
  - $\texttt{Student}(\texttt{john})$
  - $\texttt{ENROLLED}(\texttt{john}, \texttt{cs415})$
  - $(\texttt{Student} \sqcup \texttt{Professor})(\texttt{paul})$

Semantics

We consider the descriptive semantics, based on classical logics.

- An interpretation $\mathcal{I}$ *satisfies* the statement $C \sqsubseteq D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.
- An interpretation $\mathcal{I}$ *satisfies* the statement $C \doteq D$ if $C^{\mathcal{I}} = D^{\mathcal{I}}$.

An interpretation $\mathcal{I}$ is a *model* for a TBox $\mathcal{T}$ if $\mathcal{I}$ satisfies all the statements in $\mathcal{T}$.

ABox

A set $\mathcal{A}$ of assertions (membership or relationship statements) is called an ABox.

If $\mathcal{I} = (D^{\mathcal{I}}, \cdot_{\mathcal{I}})$ is an interpretation,

- $C(a)$ is satisfied by $\mathcal{I}$ if $a^{\mathcal{I}} \in C^{\mathcal{I}}$.
- $R(a, b)$ is satisfied by $\mathcal{I}$ if $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$.

An interpretation $\mathcal{I}$ is said to be a *model* of the ABox $\mathcal{A}$ if every assertion of $\mathcal{A}$ is satisfied by $\mathcal{I}$.

The ABox $\mathcal{A}$ is said to be *satisfiable* if it admits a model.

Semantics

An interpretation $\mathcal{I} = (D^{\mathcal{I}}, \cdot_{\mathcal{I}})$ is said to be a *model* of a knowledge base $(\mathcal{T}, \mathcal{A})$ if every axiom of the knowledge base is satisfied by $\mathcal{I}$.

A knowledge base $(\mathcal{T}, \mathcal{A})$ is said to be *satisfiable* if it admits a model.

Logical Implication

$(\mathcal{T}, \mathcal{A}) \models \varphi$ if every model of $(\mathcal{T}, \mathcal{A})$ is a model of φ

Example 1:

- TBox: $\mathcal{T}$
  - $\texttt{Student} \doteq \texttt{Person} \sqcap \exists\texttt{NAME.String} \sqcap \exists\texttt{ADDRESS.String} \sqcap \exists\texttt{ENROLLED.Course}$
  - $\texttt{Student} \sqsubseteq \exists\texttt{ENROLLED.Course}$
  - $\exists\texttt{TEACHES.Course} \sqsubseteq \neg\texttt{Undergrad} \sqcup \texttt{Professor}$
- ABox: $\mathcal{A} = \emptyset$

$(\mathcal{T}, \mathcal{A}) \overset{?}{\models} \texttt{Student} \sqsubseteq \texttt{Person}$

Logical Implication

$(\mathcal{T}, \mathcal{A}) \models \varphi$ if every model of Σ is a model of φ

Example 2:

TBox: $\mathcal{T}$

$\exists \mathtt{TEACHES}.\mathtt{Course} \sqsubseteq \neg \mathtt{Undergrad} \sqcup \mathtt{Professor}$

ABox: $\mathcal{A}$

$\mathtt{TEACHES}(\mathtt{john}, \mathtt{cs415}), \mathtt{Course}(\mathtt{cs415}),$

$\mathtt{Undergrad}(\mathtt{john})$

$(\mathcal{T}, \mathcal{A}) \models \mathtt{Professor}(\mathtt{john})$

Logical Implication

TBox: $\mathcal{T}$

$\exists\texttt{TEACHES}.\texttt{Course} \sqsubseteq$

$\neg\texttt{Undergrad} \sqcup \texttt{Professor}$

ABox: $\mathcal{A}$

$\texttt{TEACHES}(\texttt{john}, \texttt{cs415}), \texttt{Course}(\texttt{cs415}),$

$\texttt{Undergrad}(\texttt{john})$

$$(\mathcal{T}, \mathcal{A}) \overset{?}{\models} \texttt{Professor}(\texttt{john})$$

$$(\mathcal{T}, \mathcal{A}) \overset{?}{\models} \neg\texttt{Professor}(\texttt{john})$$

Reasoning Problems

- **Concept Satisfiability**

  $(\mathcal{T}, \mathcal{A}) \not\models C \equiv \bot$ $\quad\quad$ $\texttt{Student} \sqcap \neg\texttt{Person}$

  the problem of checking whether C is satisfiable w.r.t. Σ, i.e. whether there exists a model $\mathcal{I}$ of Σ such that $C^{\mathcal{I}} \neq \emptyset$

- **Subsumption**

  $(\mathcal{T}, \mathcal{A}) \models C \sqsubseteq D$ $\quad\quad$ $\texttt{Student} \sqsubseteq \texttt{Person}$

  the problem of checking whether C is subsumed by D w.r.t. Σ, i.e. whether $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ in every model $\mathcal{I}$ of $(\mathcal{T}, \mathcal{A})$

- **Satisfiability**

  $(\mathcal{T}, \mathcal{A}) \not\models \text{false}$ $\quad\quad$ $\texttt{Student} \doteq \neg\texttt{Person}$

  the problem of checking whether $(\mathcal{T}, \mathcal{A})$ is satisfiable, i.e. whether it has a model

- **Instance Checking**

  $(\mathcal{T}, \mathcal{A}) \models C(a)$ $\quad\quad$ $\texttt{Professor}(\texttt{john})$

  the problem of checking whether the assertion $C(a)$ is satisfied in every model of $(\mathcal{T}, \mathcal{A})$

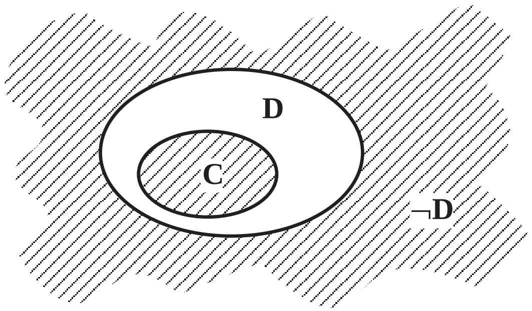
Reduction to concept satisfiability

- **Concept Satisfiability**

  $(\mathcal{T}, \mathcal{A}) \not\models C \equiv \bot \quad \leftrightarrow$
  $\mathcal{T} \cup \mathcal{A} \cup \{C(x)\}$ has a model

- **Subsumption**

  $(\mathcal{T}, \mathcal{A}) \models C \sqsubseteq D \quad \leftrightarrow$
  $(\mathcal{T}, \mathcal{A}) \models C \sqcap \neg D \equiv \bot \quad \leftrightarrow$
  $(\mathcal{T}, \mathcal{A}) \cup \{(C \sqcap \neg D)(x)\}$ has no models



- **Instance Checking**

  $(\mathcal{T}, \mathcal{A}) \models C(a) \quad \leftrightarrow$
  $(\mathcal{T}, \mathcal{A}) \cup \{\neg C(a)\}$ has no models

Other reasoning problems

Classification

- Given a concept C and a TBox T, for all concepts D of T determine whether D subsumes C, or D is subsumed by C.
- Intuitively, this amounts to finding the "right place" for C in the taxonomy implicitly present in T.
- *Classification* is the task of inserting new concepts in a taxonomy. It is *sorting* in partial orders.

Goal

- Prove decidability of description logic
- Give efficient decision procedures

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- Give efficient decision procedures

$\mathcal{ALC}$: Express it as a multi-modal logic

$\mathcal{ALC}$ as a multi-modal logic

We translate every concept C of $\mathcal{ALC}$ into a formula F_C in a many-modal logic which contains modal operators

$$\Box_R, \Diamond_R \quad \text{for every role } R$$

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In the translation we replace every primitive concept symbol with a propositional variable.

$$C \quad \mapsto \quad F_C := C \quad \text{if } C \text{ is a primitive concept}$$

$\mathcal{ALC}$ as a multi-modal logic

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$$
\begin{array}{lcll}
C & \mapsto & F_C := C & \text{if } C \text{ is a primitive concept} \\
C_1 \sqcap C_2 & \mapsto & F_{C_1 \sqcap C_2} := F_{C_1} \wedge F_{C_2} & \\
C_1 \sqcup C_2 & \mapsto & F_{C_1 \sqcup C_2} := F_{C_1} \vee F_{C_2} & \\
\neg C & \mapsto & F_{\neg C} := \neg F_C & \\
\forall R.C & \mapsto & F_{\forall R.C} := \Box_R F_C & \\
\exists R.C & \mapsto & F_{\exists R.C} := \Diamond_R F_C &
\end{array}
$$

$\mathcal{ALC}$ as a multi-modal logic

An interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where

$$C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$$

$$R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$$

clearly corresponds to a (multi-modal) Kripke structure
$\mathcal{K} = (S, \{\rho_R\}_{R \in N_R}, I)$ where

- $S = \Delta^{\mathcal{I}}$
- $\rho_R = R^{\mathcal{I}}$
- $I : \Pi \times S \to \{0, 1\}$ (where $\Pi = N_C$) is defined by:
  $I(C, x) = 1$ iff $x \in C^{\mathcal{I}}$

$\mathcal{ALC}$ as a multi-modal logic

Lemma. For every $\mathcal{ALC}$ concept C and every interpretation $\mathcal{I}$ we have:

$$C^{\mathcal{I}} = \{d \in \Delta^{\mathcal{I}} \mid (\mathcal{K}, d) \models F_C\}.$$

Proof: Structural induction

If $C \in N_C$ the result follows from the way the valuation of $\mathcal{K}$ is defined.

For the induction step we here only consider the case $C = \forall R.C_1$
Induction hypothesis (IH): property holds for C_1.

$$\begin{aligned}
&\{d \in \Delta^{\mathcal{I}} \mid (\mathcal{K}, d) \models F_C\} = \{d \in \Delta^{\mathcal{I}} \mid (\mathcal{K}, d) \models F_{\forall R.C_1}\} && = \\
&\{d \in \Delta^{\mathcal{I}} \mid (\mathcal{K}, d) \models \Box_R F_{C_1}\} && \stackrel{IH}{=} \\
&\{d \in \Delta^{\mathcal{I}} \mid \text{ for all } e \text{ with } R(d, e) \text{ we have } (\mathcal{K}, e) \models F_{C_1}\} && = \\
&\{d \in \Delta^{\mathcal{I}} \mid \text{ for all } e \text{ with } R(d, e) \text{ we have } e \in C_1^{\mathcal{I}}\} && = (\forall R.C_1)^{\mathcal{I}} = C^{\mathcal{I}}
\end{aligned}$$

$\mathcal{ALC}$ as a multi-modal logic

Lemma There exists an interpretation $\mathcal{I}$ and a $d \in \Delta^{\mathcal{I}}$ such that $d \in C^{\mathcal{I}}$ iff F_C is satisfiable in the multi-modal logic.

Proof Immediate consequence of the previous lemma.