Non-classical logics

Lecture 10: Infinitely-valued logics (Part 1)

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Łukasiewicz logics

Łukasiewicz logics

$\mathcal{L}_n,\ n \in \mathbb{N}$	$W_n = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, 1\}$
$\mathcal{L}_{\aleph_0}$	$W_{\aleph_0} = [0,1] \cap \mathbb{Q}$
$\mathcal{L}_{\aleph_1}$	$W_{\aleph_1} = [0,1]$

Logical operations: $\vee, \wedge, \neg, \Rightarrow$

- $\vee = \max$
- $\wedge = \min$
- $\neg x = 1 - x$
- $x \Rightarrow y = \min(1, 1 - x + y)$

Łukasiewicz logics

Łukasiewicz implication $x \Rightarrow_{Ł_n} y = \min(1, 1 - x + y)$

$\mathcal{L}_n$

$\Rightarrow$	0	$\frac{1}{n-1}$	$\frac{2}{n-1}$	$\dots$	$\frac{n-2}{n-1}$	1
0	1	1	1	$\dots$	1	1
$\frac{1}{n-1}$	$\frac{n-2}{n-1}$	1	1	$\dots$	1	1
$\frac{2}{n-1}$	$\frac{n-3}{n-1}$	$\frac{n-2}{n-1}$	1	$\dots$	1	1
$\dots$						
1	0	$\frac{1}{n-1}$	$\frac{2}{n-1}$	$\dots$	$\frac{n-2}{n-1}$	1

Łukasiewicz logics

Theorems.

1. For $n, m \in \mathbb{N}$, s.t. $(m-1)|(n-1)$, we have Tautologies($\mathcal{L}_n$) $\subseteq$ Tautologies($\mathcal{L}_m$)

2. Tautologies($\mathcal{L}_{\aleph_0}$) $=$ Tautologies($\mathcal{L}_{\aleph_1}$)

3. Tautologies($\mathcal{L}_{\aleph_0}$) $= \bigcap\{$Tautologies($\mathcal{L}_n$) $\mid n \geq 2, n \in \mathbb{N}\}$

Proofs

Theorem. For $n, m \in \mathbb{N}$, s.t. $(m-1)|(n-1)$, we have Tautologies($\mathcal{L}_n$) $\subseteq$ Tautologies($\mathcal{L}_m$)

Proof

Assume $(m-1)|(n-1)$. Then $W_m \subseteq W_n$. Assume $F \in$ Tautologies($\mathcal{L}_n$). Then F evaluates to 1 under every valuation into W_n, hence also under every valuation into W_m, so $F \in$ Tautologies($\mathcal{L}_m$)

Proofs

Theorem. For $n, m \in \mathbb{N}$, s.t. $(m-1)|(n-1)$, we have Tautologies($\mathcal{L}_n$) $\subseteq$ Tautologies($\mathcal{L}_m$)

Remark: the converse also holds

If Tautologies($\mathcal{L}_n$) $\subseteq$ Tautologies($\mathcal{L}_m$) then $(m-1)|(n-1)$.

(This will be discussed in the next exercise session.)

Proofs

Theorem.
$\text{Tautologies}(\mathcal{L}_{\aleph_0}) = \text{Tautologies}(\mathcal{L}_{\aleph_1})$

Proof.
$" \supseteq "$: Since $[0,1] \cap \mathbb{Q} \subseteq [0,1]$, it is clear that
$\text{Tautologies}(\mathcal{L}_{\aleph_1}) \subseteq \text{Tautologies}(\mathcal{L}_{\aleph_0})$

Proofs

Theorem.
Tautologies($\mathcal{L}_{\aleph_0}$) = Tautologies($\mathcal{L}_{\aleph_1}$)

Proof.
" $\subseteq$ " : Let $F \in$ Tautologies($\mathcal{L}_{\aleph_0}$). Then for every assignment of values in $[0,1] \cap \mathbb{Q}$ to the propositional variables $\{P_1, \ldots, P_n\}$ of F evaluates to 1.

We can associate a function $f_F : [0,1]^n \longrightarrow [0,1]$ with F which is defined as follows:
For all $(x_1, \ldots, x_n) \in [0,1]^n$ let $\mathcal{A} : \{P_1, \ldots, P_n\} \longrightarrow [0,1]$ be defined by $\mathcal{A}(P_i) = x_i$.
We define $f_F(x_1, \ldots, x_n) := \mathcal{A}(F)$

It can be proved by structural induction that f_F is a continuous function.

Let $(a_1, \ldots, a_n) \in [0,1]^n$. It is now sufficient to choose sequences of rational numbers converging to $a_1, \ldots, a_n$ respectively. $f_F(a_1, \ldots, a_n)$ is the limit of the sequence defined this way, hence its value is 1.

Proofs

$\text{Tautologies}(\mathcal{L}_{\aleph_0}) = \bigcap\{\text{Tautologies}(\mathcal{L}_n) \mid n \geq 2, n \in \mathbb{N}\}$

Proof.

" $\subseteq$ " : Follows from the fact that $W_n \subseteq [0,1] \cap \mathbb{Q}$ for every $n \in \mathbb{N}$.

Proofs

$$\text{Tautologies}(\mathcal{L}_{\aleph_0}) = \bigcap\{\text{Tautologies}(\mathcal{L}_n) \mid n \geq 2, n \in \mathbb{N}\}$$

Proof. " $\supseteq$ "

Let F be a formula with prop. variables $\{P_1, \ldots, P_k\}$ s.t. $F \notin \text{Tautologies}(\mathcal{L}_{\aleph_0})$. Then there exists $\mathcal{A} : \{P_1, \ldots, P_k\} \to [0, 1] \cap \mathbb{Q}$ s.t. $\mathcal{A}(F) \neq 1$.

Assume that $\mathcal{A}(P_1) = \frac{q_1}{p_1}, \ldots, \mathcal{A}(P_k) = \frac{q_k}{p_k}$

Let $m = \text{lcm}(p_1, \ldots, p_k)$. Then it is easy to see that $\mathcal{A}(P_i) \in W_{m+1}$ for all $1 \leq i \leq k$.

We thus constructed a valuation $\mathcal{A} : \{P_1, \ldots, P_k\} \to W_m$ such that $\mathcal{A}(F) \neq 1$. Hence, $F \notin \text{Tautologies}(\mathcal{L}_m)$, so

$$F \notin \bigcap\{\text{Tautologies}(\mathcal{L}_n) \mid n \geq 2, n \in \mathbb{N}\}$$

"Fuzzy" logics

$W = [0, 1]$

Question: How to define conjunction?

Answer: Desired conditions

$f : [0, 1]^2 \longrightarrow [0, 1]$ such that:

- f associative and commutative
- for all $0 \leq A \leq B \leq 1$ and all $0 \leq C \leq 1$ we have $f(A, C) \leq f(B, C)$
- for all $0 \leq C \leq 1$ we have $f(C, 1) = C$.

Definition A function with the properties above is called a t-norm.

Examples of t-norms

Gödel t-norm	$f_G(x, y) = \min(x, y)$
Łukasiewicz t-norm	$f_{Ł}(x, y) = \max(0, x + y - 1)$
Product t-norm	$f_P(x, y) = x \cdot y$

Left-continuous t-norm

Definition. A t-norm f is left-continuous if for every $x, y \in [0, 1]$ and every sequence $\{x_n\}_{n \in \mathbb{N}}$ with $0 \leq x_n \leq x$ and $lim_{n \to \infty} x_n = x$ we have $lim_{n \to \infty} f(x_n, y) = f(x, y)$.

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The following t-norms are left continuous:

Gödel t-norm	$f_G(x, y) = \min(x, y)$
Łukasiewicz t-norm	$f_Ł(x, y) = \max(0, x + y - 1)$
Product t-norm	$f_P(x, y) = x \cdot y$

Left continuous t-norms

With every left continuous t-norm f we can associate the following operations:

- $x \circ_f y = f(x, y)$
- $x \oplus_f y = 1 - f(1 - x, 1 - y)$
- $x \Rightarrow_f y = \max\{z \mid f(x, z) \leq y\}$
- $\neg_f x = x \Rightarrow_f 0$

Remark: Left continuity ensures that $\max\{z \mid f(x, z) \leq y\}$ exists.

Validity: $D = \{1\}$

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Łukasiewicz t-norm

$$x \circ_{Ł} y = \max(0, x + y - 1)$$

$$x \oplus_{Ł} y = 1 - \max(0, 1 - x - y)$$

$$x \Rightarrow_f y = \min(1, 1 - x + y)$$

$$\neg x = \min(1, 1 - x) = 1 - x$$

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Łukasiewicz t-norm

$$x \circ_Ł y = \max(0, x + y - 1) \qquad x \wedge_Ł y = x \circ_Ł (x \Rightarrow y)$$

$$x \oplus_Ł y = 1 - \max(0, 1 - x - y) \qquad x \vee_Ł y = \neg_Ł((\neg_Ł x) \wedge_Ł (\neg_Ł y))$$

$$x \Rightarrow_f y = \min(1, 1 - x + y)$$

$$\neg x = \min(1, 1 - x) = 1 - x$$

Left continuous t-norms

With every left continuous t-norm f we can associate the following operations:

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Gödel t-norm

$x \circ_G y = \min(x, y)$

$x \oplus_G y = \max(x, y)$

$$x \Rightarrow_G y = \max\{z \mid x \wedge z \leq y\} = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x > y \end{cases}$$

$$\neg_G x = \max\{z \mid x \wedge z = 0) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$