Non-classical logics

Lecture 11: Infinitely-valued logics (Part 2)

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Last time:

Łukasiewicz logics

$\mathcal{L}_n,\ n \in \mathbb{N}$	$W_n = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, 1\}$
$\mathcal{L}_{\aleph_0}$	$W_{\aleph_0} = [0,1] \cap \mathbb{Q}$
$\mathcal{L}_{\aleph_1}$	$W_{\aleph_1} = [0,1]$

Logical operations: $\vee, \wedge, \neg, \Rightarrow$

- $\vee = \max$
- $\wedge = \min$
- $\neg x = 1 - x$
- $x \Rightarrow y = \min(1, 1 - x + y)$

Łukasiewicz logics

Theorems.

1. For $n, m \in \mathbb{N}$, s.t. $(m-1)|(n-1)$, we have Tautologies$(\mathcal{L}_n) \subseteq$ Tautologies$(\mathcal{L}_m)$
2. Tautologies$(\mathcal{L}_{\aleph_0}) =$ Tautologies$(\mathcal{L}_{\aleph_1})$
3. Tautologies$(\mathcal{L}_{\aleph_0}) = \bigcap\{$Tautologies$(\mathcal{L}_n) \mid n \geq 2, n \in \mathbb{N}\}$

"Fuzzy" logics

$W = [0, 1]$

t-norm:
$f : [0, 1]^2 \longrightarrow [0, 1]$ such that:

- f associative and commutative
- for all $0 \leq A \leq B \leq 1$ and all $0 \leq C \leq 1$ we have $f(A, C) \leq f(B, C)$
- for all $0 \leq C \leq 1$ we have $f(C, 1) = C$.

Gödel t-norm	$f_G(x, y) = \min(x, y)$
Łukasiewicz t-norm	$f_{Ł}(x, y) = \max(0, x + y - 1)$
Product t-norm	$f_P(x, y) = x \cdot y$

Left-continuous t-norm

Definition. A t-norm f is left-continuous if for every $x, y \in [0, 1]$ and every sequence $\{x_n\}_{n \in \mathbb{N}}$ with $0 \leq x_n \leq x$ and $lim_{n \to \infty} x_n = x$ we have $lim_{n \to \infty} f(x_n, y) = f(x, y)$.

The following t-norms are left continuous:

Gödel t-norm	$f_G(x, y) = \min(x, y)$
Łukasiewicz t-norm	$f_Ł(x, y) = \max(0, x + y - 1)$
Product t-norm	$f_P(x, y) = x \cdot y$

Left continuous t-norms

With every left continuous t-norm f we can associate the following operations:

- $x \circ_f y = f(x, y)$
- $x \oplus_f y = 1 - f(1 - x, 1 - y)$
- $x \Rightarrow_f y = \max\{z \mid f(x, z) \leq y\}$
- $\neg_f x = x \Rightarrow_f 0$

Łukasiewicz t-norm

$$x \circ_{Ł} y = \max(0, x + y - 1) \qquad x \wedge_{Ł} y = x \circ_{Ł} (x \Rightarrow y)$$

$$x \oplus_{Ł} y = 1 - \max(0, 1 - x - y) \qquad x \vee_{Ł} y = \neg_{Ł}((\neg_{Ł} x) \wedge_{Ł} (\neg_{Ł} y))$$

$$x \Rightarrow_f y = \min(1, 1 - x + y)$$

$$\neg x = \min(1, 1 - x) = 1 - x$$

Left continuous t-norms

With every left continuous t-norm f we can associate the following operations:

- $x \circ_f y = f(x, y)$
- $x \oplus_f y = 1 - f(1 - x, 1 - y)$
- $x \Rightarrow_f y = \max\{z \mid f(x, z) \leq y\}$
- $\neg_f x = x \Rightarrow_f 0$

Gödel t-norm

$$x \circ_G y = \min(x, y)$$

$$x \oplus_G y = \max(x, y)$$

$$x \Rightarrow_G y = \max\{z \mid x \wedge z \leq y\} = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x > y \end{cases}$$

$$\neg_G x = \max\{z \mid x \wedge z = 0) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$

Checking validity of formulae in fuzzy logics

Given: F formula in a t-norm based fuzzy logic formed with the operations $\{\circ, \oplus, \neg, \Rightarrow\}$ (and also $\vee, \wedge$ if definable)

Task: Check whether F is valid (a tautology)
i.e. whether for all $\mathcal{A} : X \rightarrow [0, 1]$, $\mathcal{A}(F) = 1$

Idea:
Assume that there exists $\mathcal{A} : X \rightarrow [0, 1]$ such that $\mathcal{A}(F) \neq 1$.
Derive a contradiction.

Let $P_1, \dots, P_n$ be the propositional variables which occur in F.

Check whether $\exists x_1, \dots, x_n F(x_1, \dots, x_m) \neq 1$ is satisfiable in
$\mathcal{A} = ([0, 1], \{\circ_f, \oplus_f, \neg_f, \rightarrow_f, \leftrightarrow_f\})$.

Example 1: Łukasiewicz logic Ł $= \mathcal{L}_{\alpha_1}$

F $\mathcal{F}$-formula, where $\mathcal{F} = \{\vee, \wedge, \circ, \neg, \rightarrow, \leftrightarrow\}$.

Let $P_1, \dots, P_n$ be the propositional variables which occur in F.

Check whether $\exists x_1, \dots, x_n F(x_1, \dots, x_m) \neq 1$ is satisfiable in

$$[0,1]_{Ł} = ([0,1], \{\vee, \wedge, \circ, \neg, \rightarrow\})$$

where $\vee, \wedge, \neg, \rightarrow, \leftrightarrow$ are the operations induced by the t-norm $f_{Ł}(x,y) = \max(0, x+y-1)$, i.e.:

$(\mathrm{Def}_{\circ_{Ł}})$	$x{+}y{<}1 \rightarrow x{\circ}y{=}0$	$x{+}y{\geq}1 \rightarrow x{\circ}y{=}x{+}y{-}1$
$(\mathrm{Def}_{\vee})$	$x{\leq}y \rightarrow x{\vee}y{=}y$	$x{>}y \rightarrow x{\vee}y{=}x$
$(\mathrm{Def}_{\wedge})$	$x{\leq}y \rightarrow x{\wedge}y = x$	$x{>}y \rightarrow x{\wedge}y = y$
$(\mathrm{Def}_{\Rightarrow_{Ł}})$	$x{\leq}y \rightarrow x{\Rightarrow}y{=}1$	$x{>}y \rightarrow x{\Rightarrow}y = 1{-}x{+}y$
$(\mathrm{Def}_{\neg_{Ł}})$	$\neg x = 1 - x$	

Example 1: Łukasiewicz logic Ł $= \mathcal{L}_{\alpha_1}$

F $\mathcal{F}$-formula, where $\mathcal{F} = \{\vee, \wedge, \circ, \neg, \rightarrow, \leftrightarrow\}$.

Remark: The following are equivalent:

(1) $F(x_1, \dots, x_m) \neq 1$ is satisfiable in $[0,1]_Ł = ([0,1], \{\vee, \wedge, \circ, \neg, \rightarrow\})$, where $\vee, \wedge, \neg, \rightarrow, \leftrightarrow$ are the operations induced by the t-norm $f_Ł$

(2) $\text{Def}_Ł \wedge F(x_1, \dots, x_m) \neq 1$ satisfiable in $[0,1]$.

$(\text{Def}_{\circ_Ł})$	$x+y<1 \rightarrow x \circ y = 0$	$x+y \geq 1 \rightarrow x \circ y = x+y-1$
$(\text{Def}_{\vee})$	$x \leq y \rightarrow x \vee y = y$	$x > y \rightarrow x \vee y = x$
$(\text{Def}_{\wedge})$	$x \leq y \rightarrow x \wedge y = x$	$x > y \rightarrow x \wedge y = y$
$(\text{Def}_{\Rightarrow_Ł})$	$x \leq y \rightarrow x \Rightarrow y = 1$	$x > y \rightarrow x \Rightarrow y = 1 - x + y$
$(\text{Def}_{\neg_Ł})$	$\neg x = 1 - x$	

Example

To show: $((x \Rightarrow 0) \Rightarrow 0) \Rightarrow (x \vee y)$ is a tautology

New task: $\text{Def}_{Ł} \wedge \underbrace{((x \Rightarrow 0) \Rightarrow 0) \Rightarrow (x \vee y) \neq 1}_{G_1}$ unsatisfiable

where	$(\text{Def}_{\vee})$	$x \leq y \rightarrow x \vee y = y$	$x > y \rightarrow x \vee y = x$
	$(\text{Def}_{\wedge})$	$x \leq y \rightarrow x \wedge y = x$	$x > y \rightarrow x \wedge y = y$
	$(\text{Def}_{\circ_{Ł}})$	$x + y < 1 \rightarrow x \circ y = 0$	$x + y \geq 1 \rightarrow x \circ y = x + y - 1$
	$(\text{Def}_{\Rightarrow_{Ł}})$	$x \leq y \rightarrow x \Rightarrow y = 1$	$x > y \rightarrow x \Rightarrow y = 1 - x + y$

Example

New task: $\text{Def}_{Ł} \wedge \underbrace{((x \Rightarrow 0) \Rightarrow 0) \Rightarrow (x \vee y) \neq 1}_{G_1}$ unsatisfiable

where	$(\text{Def}_\vee)$	$x \leq y \rightarrow x \vee y = y$	$x > y \rightarrow x \vee y = x$
	$(\text{Def}_\wedge)$	$x \leq y \rightarrow x \wedge y = x$	$x > y \rightarrow x \wedge y = y$
	$(\text{Def}_{\circ_Ł})$	$x + y < 1 \rightarrow x \circ y = 0$	$x + y \geq 1 \rightarrow x \circ y = x + y - 1$
	$(\text{Def}_{\Rightarrow_Ł})$	$x \leq y \rightarrow x \Rightarrow y = 1$	$x > y \rightarrow x \Rightarrow y = 1 - x + y$

1. Rename subterms starting with Ł-operators and expand definitions:

$p = x \Rightarrow 0$	$s \neq 1$	$x \leq 0 \rightarrow x \Rightarrow 0 = 1$	$x > 0 \rightarrow x \Rightarrow 0 = 1 - x + 0$
$q = p \Rightarrow 0$		$p \leq 0 \rightarrow p \Rightarrow 0 = 1$	$p > 0 \rightarrow p \Rightarrow 0 = 1 - p + 0$
$r = x \vee y$		$q \leq r \rightarrow q \Rightarrow r = 1$	$q > r \rightarrow q \Rightarrow r = 1 - q + r$
$s = q \Rightarrow r$		$x \leq y \rightarrow x \vee y = y$	$x > y \rightarrow x \vee y = x$

Example

New task: $\text{Def}_{Ł} \wedge \underbrace{((x \Rightarrow 0) \Rightarrow 0) \Rightarrow (x \vee y) \neq 1}_{G_1}$ unsatisfiable

where	$(\text{Def}_{\vee})$	$x \leq y \to x \vee y = y$	$x > y \to x \vee y = x$
	$(\text{Def}_{\wedge})$	$x \leq y \to x \wedge y = x$	$x > y \to x \wedge y = y$
	$(\text{Def}_{\circ_{Ł}})$	$x + y < 1 \to x \circ y = 0$	$x + y \geq 1 \to x \circ y = x + y - 1$
	$(\text{Def}_{\Rightarrow_{Ł}})$	$x \leq y \to x \Rightarrow y = 1$	$x > y \to x \Rightarrow y = 1 - x + y$

2. Replace terms starting with Ł-operations; SAT checking in [0, 1]

$p = x \Rightarrow 0$	$s \neq 1$	$x \leq 0 \to p = 1$	$x > 0 \to p = 1 - x + 0$
$q = p \Rightarrow 0$		$p \leq 0 \to q = 1$	$p > 0 \to q = 1 - p + 0$
$r = x \vee y$		$q \leq r \to s = 1$	$q > r \to s = 1 - q + r$
$s = q \Rightarrow r$		$x \leq y \to r = y$	$x > y \to r = x$

Reduction to checking constraints over [0, 1]

Reduction to checking satisfiability in [0, 1] of constraints in linear arithmetic (implications of LA expressions).

NP complete [Sonntag'85]

Similar techniques can be used also for Gödel logics (with the Gödel t-norm).

This method was first described (in a slightly more general context) in:

Viorica Sofronie-Stokkermans and Carsten Ihlemann,
"Automated reasoning in some local extensions of ordered structures."
Proceedings of ISMVL'07, IEEE Press, paper 1, 2007.

and (with full proofs) in

Viorica Sofronie-Stokkermans and Carsten Ihlemann,
"Automated reasoning in some local extensions of ordered structures."
Journal of Multiple-Valued Logics and Soft Computing
(Special issue dedicated to ISMVL'07) 13 (4-6), 397-414, 2007.

Example 1: Gödel logic

F $\mathcal{F}$-formula, where $\mathcal{F} = \{\vee, \wedge, \neg, \rightarrow, \leftrightarrow\}$.

Let $P_1, \ldots, P_n$ be the propositional variables which occur in F.

Check whether $\exists x_1, \ldots, x_n F(x_1, \ldots, x_m) \neq 1$ is satisfiable in

$$[0, 1]_G = ([0, 1], \{\vee, \wedge, \circ, \neg, \rightarrow\})$$

where $\vee, \wedge, \neg, \rightarrow, \leftrightarrow$ are the operations induced by the t-norm $f_{\mathsf{G}}(x, y) = \min(x, y)$, i.e.:

$$\begin{array}{rll}
(\mathsf{Def}_\circ) = (\mathsf{Def}_\wedge) & x \leq y \rightarrow x \wedge y = x & x > y \rightarrow x \wedge y = y \\
(\mathsf{Def}_\vee) & x \leq y \rightarrow x \vee y = y & x > y \rightarrow x \vee y = x \\
(\mathsf{Def}_\Rightarrow) & x \leq y \rightarrow x \Rightarrow y = 1 & x > y \rightarrow x \Rightarrow y = y \\
(\mathsf{Def}_\neg) & x = 0 \rightarrow \neg x = 1 & x > 0 \rightarrow \neg x = 0
\end{array}$$

Example

Check whether $((x \Rightarrow 0) \Rightarrow 0) \Rightarrow (x \vee y)$ is a tautology in the Gödel logic.
New task: $\text{Def}_G \wedge \underbrace{((x \Rightarrow 0) \Rightarrow 0) \Rightarrow (x \vee y) \neq 1}_{G_1}$ satisfiable?

where $(\text{Def}_\circ) = (\text{Def}_\wedge) \quad x \leq y \rightarrow x \wedge y = x \qquad x > y \rightarrow x \wedge y = y$

$(\text{Def}_\vee) \quad x \leq y \rightarrow x \vee y = y \qquad x > y \rightarrow x \vee y = x$

$(\text{Def}_\Rightarrow) \quad x \leq y \rightarrow x \Rightarrow y = 1 \qquad x > y \rightarrow x \Rightarrow y = y$

$(\text{Def}_\neg) \quad x = 0 \rightarrow \neg x = 1 \qquad x > 0 \rightarrow \neg x = 0$

Example

New task: $\mathrm{Def}_G \wedge \underbrace{((x \Rightarrow 0) \Rightarrow 0) \Rightarrow (x \vee y) \neq 1}_{G_1}$ satisfiable?

where $(\mathrm{Def}_\circ) = (\mathrm{Def}_\wedge)$ $\quad x \leq y \rightarrow x \wedge y = x \qquad x > y \rightarrow x \wedge y = y$

$(\mathrm{Def}_\vee) \quad x \leq y \rightarrow x \vee y = y \qquad x > y \rightarrow x \vee y = x$

$(\mathrm{Def}_\Rightarrow) \quad x \leq y \rightarrow x \Rightarrow y = 1 \qquad x > y \rightarrow x \Rightarrow y = y$

$(\mathrm{Def}_\neg) \quad x = 0 \rightarrow \neg x = 1 \qquad x > 0 \rightarrow \neg x = 0$

1. Rename subterms starting with Ł-operators and expand definitions:

$$
\begin{array}{l|l}
p = x \Rightarrow 0 & s \neq 1 \quad x \leq 0 \rightarrow x \Rightarrow 0 = 1 \quad x > 0 \rightarrow x \Rightarrow 0 = 0 \\
q = p \Rightarrow 0 & \qquad\quad p \leq 0 \rightarrow p \Rightarrow 0 = 1 \quad p > 0 \rightarrow p \Rightarrow 0 = 0 \\
r = x \vee y & \qquad\quad q \leq r \rightarrow q \Rightarrow r = 1 \quad q > r \rightarrow q \Rightarrow r = r \\
s = q \Rightarrow r & \qquad\quad x \leq y \rightarrow x \vee y = y \quad x > y \rightarrow x \vee y = x
\end{array}
$$

Example

New task: $\text{Def}_G \wedge \underbrace{((x \Rightarrow 0) \Rightarrow 0) \Rightarrow (x \vee y) \neq 1}_{G_1}$ satisfiable?

where $(\text{Def}_\circ) = (\text{Def}_\wedge)$ $x \leq y \rightarrow x \wedge y = x$ $x > y \rightarrow x \wedge y = y$

$(\text{Def}_\vee)$ $x \leq y \rightarrow x \vee y = y$ $x > y \rightarrow x \vee y = x$

$(\text{Def}_\Rightarrow)$ $x \leq y \rightarrow x \Rightarrow y = 1$ $x > y \rightarrow x \Rightarrow y = y$

$(\text{Def}_\neg)$ $x = 0 \rightarrow \neg x = 1$ $x > 0 \rightarrow \neg x = 0$

2. Replace terms starting with Ł-operations; SAT checking in [0, 1]

$p = x \Rightarrow 0$	$s \neq 1$	$x \leq 0 \rightarrow p = 1$	$x > 0 \rightarrow p = 0$
$q = p \Rightarrow 0$		$p \leq 0 \rightarrow q = 1$	$p > 0 \rightarrow q = 0$
$r = x \vee y$		$q \leq r \rightarrow s = 1$	$q > r \rightarrow s = r$
$s = q \Rightarrow r$		$x \leq y \rightarrow r = y$	$x > y \rightarrow r = x$

Satisfiable (e.g. by $\beta(x)=\beta(y)=\frac{1}{2}$), so $((x \Rightarrow 0) \Rightarrow 0) \Rightarrow (x \vee y)$ not tautology in Gödel logic.

Product logic

Similar techniques can be used also for the product logic
(with the product t-norm)

$\mapsto$ non-linearity (hence higher complexity)