Non-classical logics

Lecture 6: Many-valued logics (Part 2)

Viorica Sofronie-Stokkermans

sofronie@uni-koblenz.de

Until now

- Many-valued logic (finitely-valued; infinitely-valued)

  History and Motivation

  Syntax

  Semantics

1 Syntax

- propositional variables Π
- logical operations $\mathcal{F}$

Propositional Formulas $F_{\Pi}^{\mathcal{F}}$ is the set of propositional formulas over Π defined as follows:

$$
\begin{array}{rcll}
F, G, H & ::= & c & \text{(c constant logical operator)} \\
& \mid & P, \quad P \in \Pi & \text{(atomic formula)} \\
& \mid & f(F_1, \ldots, F_n) & (f \in \mathcal{F} \text{ with arity } n)
\end{array}
$$

Semantics

We assume that a set $M = \{w_1, w_2, \ldots, w_m\}$ of truth values is given.
We assume that a subset $D \subseteq M$ of designated truth values is given.

1. Meaning of the logical operators

$f \in \mathcal{F}$ with arity $n \quad \mapsto \quad f_M : M^n \to M$ (truth tables for the operations in $\mathcal{F}$)

2. The meaning of the propositional variables

A Π-valuation is a map $\mathcal{A} : \Pi \to M$.

3. Truth value of a formula in a valuation

Given an interpretation of the operation symbols $(M, \{f_M\}_{f \in \mathcal{F}})$, any Π-valuation $\mathcal{A} : \Pi \to M$, can be extended to $\mathcal{A}^* : \Sigma\text{-formulas} \to M$.

$$\mathcal{A}^*(c) = c_M \text{(for every constant operator } c \in \mathcal{F})$$

$$\mathcal{A}^*(P) = \mathcal{A}(P)$$

$$\mathcal{A}^*(f(F_1, \ldots, F_n)) = f_M(\mathcal{A}^*(F_1), \ldots, \mathcal{A}^*(F_n))$$

For simplicity, we write $\mathcal{A}$ instead of $\mathcal{A}^*$.

Models, Validity, and Satisfiability

$M = \{w_1, \ldots, w_m\}$ set of truth values
$D \subseteq M$ set of designated truth values
$\mathcal{A} : \Pi \rightarrow M$.

F is valid in $\mathcal{A}$ ($\mathcal{A}$ is a model of F; F holds under $\mathcal{A}$):

$$\mathcal{A} \models F :\Leftrightarrow \mathcal{A}(F) \in D$$

F is valid (or is a tautology):

$$\models F :\Leftrightarrow \mathcal{A} \models F \text{ for all } \Pi\text{-valuations } \mathcal{A}$$

F is called satisfiable iff there exists an $\mathcal{A}$ such that $\mathcal{A} \models F$.

Otherwise F is called unsatisfiable (or contradictory).

The logic $\mathcal{L}_3$

Set of truth values: $M = \{1, u, 0\}$.

Designated truth values: $D = \{1\}$.

Logical operators: $\mathcal{F} = \{\vee, \wedge, \neg, \sim\}$.

Truth tables for the operators

$\vee$	0	u	1
0	0	u	1
u	u	u	1
1	1	1	1

$\wedge$	0	u	1
0	0	0	0
u	0	u	u
1	0	u	1

$v(F \wedge G) = \min(v(F), v(G))$
$v(F \vee G) = \max(v(F), v(G))$

Under the assumption that $0 < u < 1$.

Truth tables for negations

A	$\neg A$	$\sim A$	$\sim\neg A$	$\sim\sim A$	$\neg\neg A$	$\neg\sim A$
1	0	0	1	1	1	1
u	u	1	1	0	u	0
0	1	1	0	0	0	0

Translation in natural language:

$v(A) = 1$ gdw. A is true

$v(\neg A) = 1$ gdw. A is false

$v(\sim A) = 1$ gdw. A is not true

$v(\sim\neg A) = 1$ gdw. A is not false

First-order many-valued logic

1. Syntax

- non-logical symbols (domain-specific)
  $\Rightarrow$ terms, atomic formulas

- logical symbols $\mathcal{F}$, quantifiers
  $\Rightarrow$ formulae

Signature;Variables; Terms/Atoms/Formulae

Signature: $\Sigma = (\Omega, \Pi)$, where

- Ω: set of function symbols f with arity $n \geq 0$, written f/n,
- Π: set of predicate symbols p with arity $m \geq 0$, written p/m.

Variables: Countably infinite set X.

Terms: As in classical logic

Atoms: (atomic formulas) over Σ are formed according to this syntax:

$$A, B \quad ::= \quad p(s_1, ..., s_m) \quad , \; p/m \in \Pi$$

Formulae:

$\mathcal{F}$ set of logical operations; $\mathcal{Q} = \{Q_1, \ldots, Q_k\}$ set of quantifiers

$F_\Sigma(X)$ is the set of first-order formulas over Σ defined as follows:

$$\begin{array}{lcll} F, G, H & ::= & c & (c \in \mathcal{F}, \text{ constant}) \\ & \mid & A & (\text{atomic formula}) \\ & \mid & f(F_1, \ldots, F_n) & (f \in \mathcal{F} \text{ with arity } n) \\ & \mid & QxF & (Q \in \mathcal{Q} \text{ is a quantifier}) \end{array}$$

Semantics

- **Truth values; Interpretation of logical symbols** $M = \{1, \dots, m\}$ set of truth values; $D \subseteq M$ set of designated truth values.
  – Truth tables for the logical operations: $\{f_M : M^n \rightarrow M | f/n \in \mathcal{F}\}$
  – "Truth tables" for the quantifiers: $\{Q_M : \mathcal{P}(M) \rightarrow M | Q \in \mathcal{Q}\}$

- **Interpretation of non-logical variables:** M-valued Σ-structure
  $\mathcal{A} = (U,\ (f_\mathcal{A} : U^n \rightarrow U)_{f/n \in \Omega},\ (p_\mathcal{A} : U^m \rightarrow M)_{p/m \in \Pi})$
  where $U \neq \emptyset$ is a set, called the universe of $\mathcal{A}$.

- **Variable assignments:** $\beta : X \rightarrow \mathcal{A}$ and extensions to terms $\mathcal{A}(\beta) : T_\Sigma \rightarrow \mathcal{A}$ as in classical logic.

- **Truth value of a formula in $\mathcal{A}$ with respect to β** $\mathcal{A}(\beta) : \mathsf{F}_\Sigma(X) \rightarrow M$ is defined inductively as follows:

$$
\begin{aligned}
\mathcal{A}(\beta)(c) &= c_M \\
\mathcal{A}(\beta)(p(s_1, \dots, s_n)) &= p_\mathcal{A}(\mathcal{A}(\beta)(s_1), \dots, \mathcal{A}(\beta)(s_n)) \in M \\
\mathcal{A}(\beta)(f(F_1, \dots, F_n)) &= f_M(\mathcal{A}(\beta)(F_1), \dots, \mathcal{A}(\beta)(F_n)) \\
\mathcal{A}(\beta)(QxF) &= Q_M(\{\mathcal{A}(\beta[x \mapsto a])(F) \mid a \in U\})
\end{aligned}
$$

First-order version of $\mathcal{L}_3$

$M = \{0, u, 1\}, \quad D = \{1\}$

$\mathcal{F} = \{\vee, \wedge, \neg, \sim\}$; truth values as the propositional version

$\mathcal{Q} = \{\forall, \exists\}$

$$\forall_M(S) = \begin{cases} 1 & \text{if } S = \{1\} \\ 0 & \text{if } 0 \in S \\ u & \text{otherwise} \end{cases} \qquad \exists_M(S) = \begin{cases} 1 & \text{if } 1 \in S \\ 0 & \text{if } S = \{0\} \\ u & \text{otherwise} \end{cases}$$

$\mathcal{A}(\beta)(\forall x F(x)) = 1$ *iff* for all $a \in U_\mathcal{A}$, $\mathcal{A}(\beta[x \mapsto a])(F(x)) = 1$

$\mathcal{A}(\beta)(\forall x F(x)) = 0$ *iff* for some $a \in U_\mathcal{A}$, $\mathcal{A}(\beta[x \mapsto a])(F(x)) = 0$

$\mathcal{A}(\beta)(\forall x F(x)) = u$ otherwise

$\mathcal{A}(\beta)(\exists x F(x)) = 1$ *iff* for some $a \in U_\mathcal{A}$, $\mathcal{A}(\beta[x \mapsto a])(F(x)) = 1$

$\mathcal{A}(\beta)(\exists x F(x)) = 0$ *iff* for all $a \in U_\mathcal{A}$, $\mathcal{A}(\beta[x \mapsto a])(F(x)) = 0$

$\mathcal{A}(\beta)(\forall x F(x)) = u$ otherwise

Models, Validity, and Satisfiability

F is valid in $\mathcal{A}$ under assignment β:

$$\mathcal{A}, \beta \models F \quad :\Leftrightarrow \quad \mathcal{A}(\beta)(F) \in D$$

F is valid in $\mathcal{A}$ ($\mathcal{A}$ is a model of F):

$$\mathcal{A} \models F \quad :\Leftrightarrow \quad \mathcal{A}, \beta \models F, \text{ for all } \beta \in X \to U_{\mathcal{A}}$$

F is valid:

$$\models F \quad :\Leftrightarrow \quad \mathcal{A} \models F, \text{ for all } \mathcal{A} \in \Sigma\text{-alg}$$

F is called satisfiable iff there exist $\mathcal{A}$ and β such that $\mathcal{A}, \beta \models F$.
Otherwise F is called unsatisfiable.

Entailment

$N \models F \;:\Leftrightarrow\;$ for all $\mathcal{A} \in \Sigma$-alg and $\beta \in X \rightarrow U_{\mathcal{A}}$:

if $\mathcal{A}(\beta)(G) \in D$, for all $G \in N$, then $\mathcal{A}(\beta)(F) \in D$.

Entailment

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Goal: Define a version of implication '$\Rightarrow$' such that

$F \models G$ iff $\models F \Rightarrow G$

Weak implication

The logical operations $\supset$ and $\equiv$ are introduced as defined operations:

Weak implication

$$F \supset G :=\sim F \vee G$$

Weak equivalence

$$F \equiv G := (F \supset G) \wedge (G \supset F)$$

$F \supset G$	1	u	0
1	1	u	0
u	1	1	1
0	1	1	1

$F \equiv G$	1	u	0
1	1	u	0
u	u	1	1
0	0	1	1

Strong implication

The logical operations $\rightarrow$ and $\leftrightarrow$ are introduced as defined operations:

Strong implication

$$F \rightarrow G := \neg F \vee G$$

Strong equivalence

$$F \leftrightarrow G := (F \rightarrow G) \wedge (G \rightarrow F)$$

$F \rightarrow G$	1	u	0
1	1	u	0
u	1	u	u
0	1	1	1

$F \leftrightarrow G$	1	u	0
1	1	u	0
u	u	u	u
0	0	u	1

Comparisons

Implications

$A \supset B$	1	u	0
1	1	u	0
u	1	1	1
0	1	1	1

$A \rightarrow B$	1	u	0
1	1	u	0
u	1	u	u
0	1	1	1

Equivalences

$A \equiv B$	1	u	0
1	1	u	0
u	u	1	1
0	0	1	1

$A \leftrightarrow B$	1	u	0
1	1	u	0
u	u	u	u
0	0	u	1

Equivalences

$A \supset B := \ \sim A \vee B$ $\qquad$ $A \rightarrow B := \neg A \vee B$

$$A \equiv B := (A \supset B) \wedge (B \supset A)$$

$$A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A)$$

$$A \approx B := (A \equiv B) \wedge (\neg A \equiv \neg B)$$

$$A \Leftrightarrow B := (A \leftrightarrow B) \wedge (\neg A \leftrightarrow \neg B)$$

$$A \text{ id } B := \ \sim\sim (A \approx B)$$

A	B	$A \equiv B$	$A \leftrightarrow B$	$A \approx B$	$A \Leftrightarrow B$	A id B
1	1	1	1	1	1	1
1	u	u	u	u	u	0
1	0	0	0	0	0	0
u	1	u	u	u	u	0
u	u	1	u	1	u	1
u	0	1	u	u	u	0
0	1	0	0	0	0	0
0	u	1	u	u	u	0
0	0	1	1	1	1	1

Some $\mathcal{L}_3$ tautologies

$\neg\neg A$ id A

$\sim\sim A \equiv A$

$\neg \sim A \equiv A$

$(A \wedge B) \vee C$ id $(A \vee C) \wedge (B \vee C)$

$(A \vee B) \wedge C$ id $(A \wedge C) \vee (B \wedge C)$

$\neg(A \vee B)$ id $\neg A \wedge \neg B$

$\neg(A \wedge B)$ id $\neg A \vee \neg B$

$\sim (A \vee B)$ id $\sim A \wedge \sim B$

$\sim (A \wedge B)$ id $\sim A \vee \sim B$

$\neg(\forall x A)$ id $\exists x \neg A$

$\neg(\exists x A)$ id $\forall x \neg A$

$\sim (\forall x A)$ id $\exists x \sim A$

$\sim (\exists x A)$ id $\forall x \sim A$

No occurrence of $\neg$

Lemma. Let F be a formula which does not contain the strong negation $\neg$. Then the following are equivalent:

(1) F is an $\mathcal{L}_3$-tautology.

(2) F is a two-valued tautology (negation is identified with $\sim$)

Proof.

"$\Rightarrow$" Every $\mathcal{L}_3$-tautology is a 2-valued tautology (the restriction of the operators $\vee, \wedge, \sim$ to $\{0, 1\}$ coincides with the Boolean operations $\vee, \wedge, \neg$).

"$\Leftarrow$" Assume that F is a two-valued tautology. Let $\mathcal{A}$ be an $\mathcal{L}_3$-structure and $\beta : X \rightarrow \mathcal{A}$ be a valuation. We construct a two-valued structure $\mathcal{A}'$ from $\mathcal{A}$, which agrees with $\mathcal{A}$ except for the fact that whenever $p_{\mathcal{A}}(\overline{x}) = u$ we define $p_{\mathcal{A}'}(\overline{x}) = 0$. Then $\mathcal{A}'(\beta)(F) = 1$. It can be proved that

$\mathcal{A}(\beta)(F) = 1 \Rightarrow \mathcal{A}'(\beta)(F) = 1$

$\mathcal{A}(\beta)(F) \in \{0, u\} \Rightarrow \mathcal{A}'(\beta)(F) = 0$.

Hence, $\mathcal{A}(\beta)(F) = 1$.

Exercises

1. **Let F be a formula which does not contain $\sim$.**
   Then F is not a tautology.

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   Proof. Take the valuation which maps all propositional variables to u.

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   Then F is not a tautology.

   Proof. Take the valuation which maps all propositional variables to u.

2. **Prove that for every term t, $\forall x q(x) \supset q(x)[t/x]$ is an $\mathcal{L}_3$-tautology.**

3. **Show that $\forall x q(x) \rightarrow q(x)[t/x]$ is not a tautology.**

Exercises

1. **Let F be a formula which does not contain $\sim$.**
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   Proof. Take the valuation which maps all propositional variables to u.

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3. **Show that $\forall x q(x) \rightarrow q(x)[t/x]$ is not a tautology.**

   Solution. $q \rightarrow q$ is not a tautology.

Exercises

4. **Which of the following statements are true?**

   If $F \equiv G$ is a tautology and F is a tautology then G is a tautology.

   If $F \equiv G$ is a tautology and F is satisfiable then G is satisfiable.

   If $F \equiv G$ is a tautology and F is a non-tautology then G is a non-tautology.

   If $F \equiv G$ is a tautology and F is two-valued then G is two-valued.

F is a non-tautology iff for every 3-valued structure, $\mathcal{A}$ and every valuation β, $\mathcal{A}(\beta)(F) \neq 1$.

F is two-valued iff for every 3-valued structure, $\mathcal{A}$ and every valuation β, $\mathcal{A}(\beta)(F) \in \{0, 1\}$.

Exercises

4. **Which of the following statements are true?**

   If $F \equiv G$ is a tautology and F is a tautology then G is a tautology.

   true

   If $F \equiv G$ is a tautology and F is satisfiable then G is satisfiable.

   If $F \equiv G$ is a tautology and F is a non-tautology then G is a non-tautology.

   If $F \equiv G$ is a tautology and F is two-valued then G is two-valued.

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   If $F \equiv G$ is a tautology and F is a non-tautology then G is a non-tautology.

   If $F \equiv G$ is a tautology and F is two-valued then G is two-valued.

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   true

   If $F \equiv G$ is a tautology and F is satisfiable then G is satisfiable.

   true

   If $F \equiv G$ is a tautology and F is a non-tautology then G is a non-tautology.

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   If $F \equiv G$ is a tautology and F is two-valued then G is two-valued.

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Exercises

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   If $F \equiv G$ is a tautology and F is a tautology then G is a tautology.

   true

   If $F \equiv G$ is a tautology and F is satisfiable then G is satisfiable.

   true

   If $F \equiv G$ is a tautology and F is a non-tautology then G is a non-tautology.

   true

   If $F \equiv G$ is a tautology and F is two-valued then G is two-valued.

   false

F is a non-tautology iff for every 3-valued structure, $\mathcal{A}$ and every valuation $\beta, \mathcal{A}(\beta)(F) \neq 1$.

F is two-valued iff for every 3-valued structure, $\mathcal{A}$ and every valuation $\beta, \mathcal{A}(\beta)(F) \in \{0, 1\}$.

Functional completeness

Definition A family $(M, \{f_M : M^n \rightarrow M\}_{f \in \mathcal{F}})$ is called functionally complete if every function $g : M^m \rightarrow M$ can be expressed in terms of the functions $\{f_M : M^n \rightarrow M \mid f \in \mathcal{F}\}$.

Definition A many-valued logic with finite set of truth values M and logical operators $\mathcal{F}$ is called functionally complete if for every function $g : M^m \rightarrow M$ there exists a propositional formula F of the logic such that for every $\mathcal{A} : \Pi \rightarrow M$

$$g(\mathcal{A}(x_1), \dots, \mathcal{A}(x_m)) = \mathcal{A}(F).$$

Example: Propositional logic

$F: \quad (P \lor Q) \land ((\neg P \land Q) \lor R)$

P	Q	R	$(P \lor Q)$	$\neg P$	$(\neg P \land Q)$	$((\neg P \land Q) \lor R)$	F
0	0	0	0	1	0	0	0
0	0	1	0	1	0	1	0
0	1	0	1	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	1	0	0	0	0
1	0	1	1	0	0	1	1
1	1	0	1	0	0	0	0
1	1	1	1	0	0	1	1

Example: Propositional logic

$F: \quad (P \vee Q) \wedge ((\neg P \wedge Q) \vee R)$

P	Q	R	$(P \vee Q)$	$\neg P$	$(\neg P \wedge Q)$	$((\neg P \wedge Q) \vee R)$	F
0	0	0	0	1	0	0	0
0	0	1	0	1	0	1	0
0	1	0	1	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	1	0	0	0	0
1	0	1	1	0	0	1	1
1	1	0	1	0	0	0	0
1	1	1	1	0	0	1	1

Example: Propositional logic

$F: \quad (P \lor Q) \land ((\neg P \land Q) \lor R)$

P	Q	R	$(P \lor Q)$	$\neg P$	$(\neg P \land Q)$	$((\neg P \land Q) \lor R)$	F
0	0	0	0	1	0	0	0
0	0	1	0	1	0	1	0
0	1	0	1	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	1	0	0	0	0
1	0	1	1	0	0	1	1
1	1	0	1	0	0	0	0
1	1	1	1	0	0	1	1

DNF: $(\neg P \land Q \land \neg R) \lor (\neg P \land Q \land R) \lor (P \land \neg Q \land R) \lor (P \land Q \land R)$

Functional completeness

Theorem. Propositional logic is functionally complete.

Proof. For every $g : \{0,1\}^m \rightarrow \{0,1\}$ let:

$F = \bigvee_{(a_1,\dots,a_m)\in\{0,1\}} (c_g(a_1, \dots, a_m) \wedge P_1^{a_1} \wedge \dots \wedge P_m^{a_m})$

where $P^a = \begin{cases} P & \text{if } a = 1 \\ \neg P & \text{if } a = 0 \end{cases}$

(Then clearly $\mathcal{A}(P)^a = 1$ iff $\mathcal{A}(P) = a$, i.e. $1^1 = 0^0 = 1; 1^0 = 0^1 = 0$.)

It can be easily checked that for every $\mathcal{A} : \{P_1, \dots, P_m\} \rightarrow \{0,1\}$ we have:

$g(\mathcal{A}(P_1), \dots, \mathcal{A}(P_m)) = \mathcal{A}(F)$.