Non-classical logics

Lecture 9: Many-valued logics (Part 5)

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Until now

- Many-valued logic (finitely-valued; infinitely-valued)

  History and Motivation

  Syntax

  Semantics

  Functional completeness

  Automated reasoning: Tableaux

  Many-valued resolution (propositional logic)

1. Propositional logic

Translation to (signed) clause form

Clause: Disjunction of signed literals

Signed literals:

truth values as signs: L^v

sets of truth values as signs: $S : L$ abbrev. for $\bigvee_{v \in S} L^v$

Example: Classical propositional logic

$F: \quad (P \vee Q) \wedge ((\neg P \wedge Q) \vee R)$

P	Q	R	$(P \vee Q)$	$\neg P$	$(\neg P \wedge Q)$	$((\neg P \wedge Q) \vee R)$	F	$\neg F$
0	0	0	0	1	0	0	0	1
0	0	1	0	1	0	1	0	1
0	1	0	1	1	1	1	1	0
0	1	1	1	1	1	1	1	0
1	0	0	1	0	0	0	0	1
1	0	1	1	0	0	1	1	0
1	1	0	1	0	0	0	0	1
1	1	1	1	0	0	1	1	0

CNF: (1) DNF of $\neg F$:

$$(\neg P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee (P \wedge Q \wedge \neg R)$$

(2) negate:

$$(P \vee Q \vee R) \wedge (P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee \neg Q \vee R)$$

Signed resolution: Propositional logic

Translation to signed clause form.

$\Psi = S{:}f(F_1, \ldots, F_n)$

$$DNF(\Psi) := \bigvee_{\substack{v_1,\ldots,v_n \in M \\ f_M(v_1,\ldots,v_n) \in S}} F_1^{v_1} \wedge \cdots \wedge F_n^{v_n}$$

$$CNF(\Psi) := \bigwedge_{\substack{v_1,\ldots,v_n \in M \\ f_M(v_1,\ldots,v_n) \notin S}} (M \setminus \{v_1\}){:}F_1 \vee \cdots \vee (M \setminus \{v_n\}){:}F_n$$

(negate $DNF(M \setminus S{:}f(F_1, \ldots, F_n))$)

Example

$\Rightarrow$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1
1	0	$\frac{1}{2}$	1

Compute CNF for $\{0\}{:}(F_1 \longrightarrow F_2)$:

DNF for $\{\frac{1}{2}, 1\}{:}(F_1 \longrightarrow F_2)$: $\displaystyle\bigvee_{\substack{v_1, v_2 \in \{0, \frac{1}{2}, 1\} \\ v_1 \Rightarrow v_2 \neq 0}} \{v_1\}{:}F_1 \wedge \{v_2\}{:}F_2$

$$
\begin{array}{lllll}
(F_1^0 \wedge F_2^0) & \vee & (F_1^0 \wedge F_2^{\frac{1}{2}}) & \vee & (F_1^0 \wedge F_2^1) \\
(F_1^{\frac{1}{2}} \wedge F_2^0) & \vee & (F_1^{\frac{1}{2}} \wedge F_2^{\frac{1}{2}}) & \vee & (F_1^{\frac{1}{2}} \wedge F_2^1) \\
(F_1^1 \wedge F_2^{\frac{1}{2}}) & \vee & (F_1^1 \wedge F_2^1) & &
\end{array}
$$

CNF for $\{0\}{:}(F_1 \longrightarrow F_2)$:

$$
\begin{array}{lllll}
(\{\frac{1}{2}, 1\}{:}F_1 \vee \{\frac{1}{2}, 1\}{:}F_2) & \wedge & (\{\frac{1}{2}, 1\}{:}F_1 \vee \{0, 1\}{:}F_2) & \wedge & (\{\frac{1}{2}, 1\}{:}F_1 \vee \{0, \frac{1}{2}\}{:}F_2) \\
(\{0, 1\}{:}F_1 \vee \{\frac{1}{2}, 1\}{:}F_2) & \wedge & (\{0, 1\}{:}F_1 \vee \{0, 1\}{:}F_2) & \wedge & (\{0, 1\}{:}F_1 \vee \{0, \frac{1}{2}\}{:}F_2) \\
(\{0, \frac{1}{2}\}{:}F_1 \vee \{0, 1\}{:}F_2) & \wedge & (\{0, \frac{1}{2}\}{:}F_1^1 \vee \{0, \frac{1}{2}\}{:}F_2^1) & &
\end{array}
$$

Example

$\Rightarrow$	0	$\frac{1}{2}$	1
0	1	1	1
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Compute CNF for $\{0\}{:}(F_1 \longrightarrow F_2)$:

DNF for $\{\frac{1}{2}, 1\}{:}(F_1 \longrightarrow F_2)$: $\displaystyle\bigvee_{\substack{v_1, v_2 \in \{0, \frac{1}{2}, 1\} \\ v_1 \Rightarrow v_2 \neq 0}} \{v_1\}{:}F_1 \wedge \{v_2\}{:}F_2$

$$= (\{0\}{:}F_1 \wedge \{0, \tfrac{1}{2}, 1\}{:}F_2) \vee (\{\tfrac{1}{2}\}{:}F_1 \wedge \{0, \tfrac{1}{2}, 1\}{:}F_2) \vee (\{1\} : F_1 \wedge \{\tfrac{1}{2}, 1\}{:}F_2)$$

$$= \{0\}{:}F_1 \vee \{\tfrac{1}{2}\}{:}F_1 \vee (\{1\}{:}F_1 \wedge \{\tfrac{1}{2}, 1\}{:}F_2)$$

CNF for $\{0\}{:}(F_1 \longrightarrow F_2)$:

$$\{\tfrac{1}{2}, 1\}{:}F_1 \;\wedge\; \{0, 1\}{:}F_1 \;\wedge\; (\{0, \tfrac{1}{2}\}{:}F_1 \vee \{0\}{:}F_2)$$

Optimization

$$\Psi = S{:}f(F_1, \ldots, F_n)$$

$$DNF(\Psi) := \bigvee_{v_1,\ldots,v_{n-1} \in M} \{v_1\}{:}F_1 \wedge \cdots \wedge \{v_{n-1}\}{:}F_{n-1} \wedge \{v_n \mid f_M(v_1, \ldots, v_n) \in S\}{:}F_n$$

$$CNF(\Psi) := \bigwedge_{v_1,\ldots,v_{n-1} \in M} (M \setminus \{v_1\}){:}F_1 \vee \cdots \vee (M \setminus \{v_{n-1}\}){:}F_{n-1} \vee \{v_n \mid f_M(v_1, \ldots, v_n) \in S\}{:}F_n$$

(negate $DNF(M \setminus S{:}f(F_1, \ldots, F_n))$)

Soundness

Signed resolution (propositional form)

$$\frac{P^{v_1} \vee C \qquad P^{v_2} \vee D}{C \vee D}$$

if $v_1 \neq v_2$

Signed factoring (propositional form)

$$\frac{C \vee P^{v} \vee P^{v}}{C \vee P^{v}}$$

Soundness and Completeness

Theorems.

(1) The signed resolution inference rule is sound.

(2) The signed resolution inference rule is complete.

Proof Completeness – Idea: Signed resolution can be simulated by a version of resolution which handles equality efficiently (superposition).

Signed resolution

$$\frac{P \approx v_1 \vee C \qquad P \approx v_2 \vee D}{(C \vee D)} \qquad \text{if } v_1 \neq v_2$$

Signed factoring

$$\frac{C \vee P \approx v \vee P \approx v}{C}$$

Completeness then follows from the completeness of this refinement of resolution. This also guarantees completeness of refinements of signed resolution with ordering and selection functions

Compact form of signed resolution

Propositional logic

Signs: sets of truth values

Resolution

$$\frac{S_1{:}P \vee C \qquad S_2{:}P \vee D}{(S_1 \cap S_2){:}P \vee C \vee D} \qquad \text{if } S_1 \cap S_2 = \emptyset$$

Simplificaton

$$\frac{C \vee \emptyset{:}P}{C}$$

Merging

$$\frac{S_1{:}P \vee S_2{:}P \vee C}{(S_1 \cup S_2){:}P \vee C}$$

First-order logic

Translation to clause form:
need to take into account also the truth tables of the quantifiers.

$S : QxF(x)$

DNF: $\bigvee_{\substack{\emptyset \neq V \subseteq M \\ Q_M(V) \in S}} (\forall x\ V : F(x) \wedge \bigwedge_{a \in V} \exists x\ \{a\} : F(x))$

CNF: computed by negating the DNF for $M \backslash S : \forall x F(x)$

CNF: $\bigwedge_{\substack{\emptyset \neq V \subseteq M \\ Q_M(V) \in (M \backslash S)}} (\exists x (M \backslash V) : F(x) \vee \bigvee_{a \in V} \forall x (M \backslash \{a\}) : F(x))$

$\mapsto$ leave out quantifiers (Skolem functions for existential quantifier)

Example

In $\mathcal{L}_3$, with truth values $M = \{0, u, 1\}$:

$\{1, u\}\forall x \; p(x)$

$$\Rightarrow \quad \bigwedge_{\substack{\emptyset \neq V \subseteq M \\ min(V) \in \{0\}}} (\exists x(M \setminus V) : F(x) \vee \bigvee_{a \in \{0\}} \forall x(M \setminus \{a\}) : F(x))$$

$$\Rightarrow \quad (\exists x\{1, u\}{:}p(x) \vee \forall x(M \setminus \{0\}){:}p(x)) \wedge$$

$$\Rightarrow \quad (\exists x\{u\}{:}p(x) \vee \forall x\{1, u\}{:}p(x) \vee \forall x\{0, u\}{:}p(x)) \wedge$$

$$\Rightarrow \quad (\exists x\{1\}{:}p(x) \vee \forall x\{1, u\}{:}p(x) \vee \forall x\{0, 1\}{:}p(x)) \wedge$$

$$\Rightarrow \quad \forall x\{1, u\}{:}p(x) \vee \forall x\{0, 1\} : p(x) \vee \forall x\{0, u\} : p(x))$$

Structure-preserving translation

In order to avoid rapid growth of the number of clauses, a structure-preserving translation to clause form is used.

Idea

$$S : F[G(x)] \quad \Rightarrow \quad S : F[P_{G(x)}(x)] \wedge \bigwedge_{a \in M} \forall x(\{a\} G(x) \leftrightarrow \{a\} : P_{G(x)}(x))$$

where $P_{G(x)}$ new predicate symbol.

$$S{:}F[\underbrace{f(F_1, \dots, F_n)}_{G}]$$

$$\Rightarrow \quad S : F[P_G] \wedge \bigwedge_{a \in M} \forall x(DNF(\{a\} : f(F_1, \dots, F_n) \leftrightarrow \{a\} : P_G)$$

Resolution for first-order clauses

Natural generalization of the resolution rule:

Signed resolution

$$\frac{L_1^{v_1} \vee C \qquad L_2^{v_2} \vee D}{(C \vee D)\sigma}$$

if $v_1 \neq v_2$, and $\sigma = \mathrm{mgu}(L_1, L_2)$

Signed factoring

$$\frac{C \vee L_1^{v} \vee L_2^{v}}{(C \vee L_1^{v})\sigma}$$

if $\sigma = \mathrm{mgu}(L_1, L_2)$

Regular logics

Many-valued logics for which an order $\leq$ exists on the sets of truth values and for which signed CNF's can be found which contain as signs only the sets

$$\uparrow i = \{j \in M \mid j \geq i\} \text{ and } \downarrow i = \{j \in M \mid j \leq i\}$$

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Many-valued logics for which an order $\leq$ exists on the sets of truth values and for which signed CNF's can be found which contain as signs only the sets

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Example

Łukasiewicz logics $\mathcal{L}_n$

- Set of truth values $M = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, 1\}$
- Logical operations: $\vee, \wedge, \neg, \Rightarrow$
  - $\vee_{Ł_n} = \max$
  - $\wedge_{Ł_n} = \min$
  - $\neg_{Ł_n} x = 1 - x$
  - $x \Rightarrow_{Ł_n} y = \min(1, 1 - x + y)$
- First-order version: $\mathcal{Q} = \{\forall, \exists\}$

Łukasiewicz logics

Łukasiewicz implication $x \Rightarrow_{Ł_n} y = \min(1, 1 - x + y)$

$\mathcal{L}_n$

$\Rightarrow$	0	$\frac{1}{n-1}$	$\frac{2}{n-1}$	$\ldots$	$\frac{n-2}{n-1}$	1
0	1	1	1	$\ldots$	1	1
$\frac{1}{n-1}$	$\frac{n-2}{n-1}$	1	1	$\ldots$	1	1
$\frac{2}{n-1}$	$\frac{n-3}{n-1}$	$\frac{n-2}{n-1}$	1	$\ldots$	1	1
$\ldots$						
1	0	$\frac{1}{n-1}$	$\frac{2}{n-1}$	$\ldots$	$\frac{n-2}{n-1}$	1

Example

$\uparrow i : (F_1 \land F_2) \mapsto (\uparrow i : F_1) \land (\uparrow i : F_2)$

$\uparrow i : (F_1 \lor F_2) \mapsto (\uparrow i : F_1) \lor (\uparrow i : F_2)$

$\uparrow i : \neg F \quad \mapsto \downarrow (1 - i) : F$

$\uparrow i : F_1 \Rightarrow F_2 \quad \mapsto \bigvee_{j \in M} (\downarrow j : F_1 \land \uparrow (i + j - 1) : F_2$

Similar for $\downarrow i : F$

signed CNFs can be obtained using the transformation rules above (and possibly negation).