# Non-classical logics

**Lecture 12:** Modal logics (Part 2)

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# Semantics of modal logic

Two classes of models have been studied so far.

- Modal algebras
- Kripke models

# Kripke Frames and Kripke Structures

Introduced by Saul Aaron Kripke in 1959.

Much less complicated and better suited to automated reasoning than modal algebras.

# Kripke Frames and Kripke Structures

**Definition.** A Kripke frame $F = (S, R)$ consists of

- a non-empty set S (of possible worlds / states)
- an accessibility relation $R \subseteq S \times S$

# Kripke Frames and Kripke Structures

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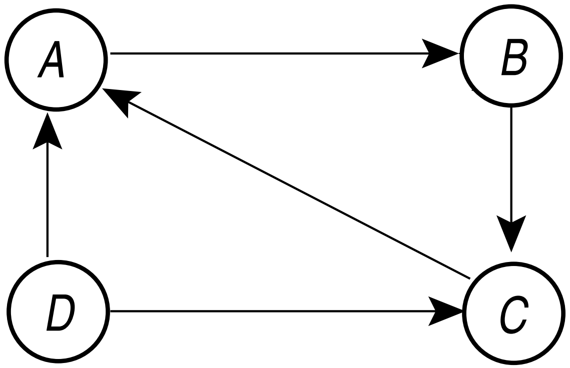
- a non-empty set S (of possible worlds / states)
- an accessibility relation $R \subseteq S \times S$

**Definition.** A Kripke structure $K = (S, R, \mathcal{I})$ consists of

- a Kripke frame $F = (S, R)$
- an interpretation $\mathcal{I} : \Pi \times S \rightarrow \{1, 0\}$

# Example of Kripke frame



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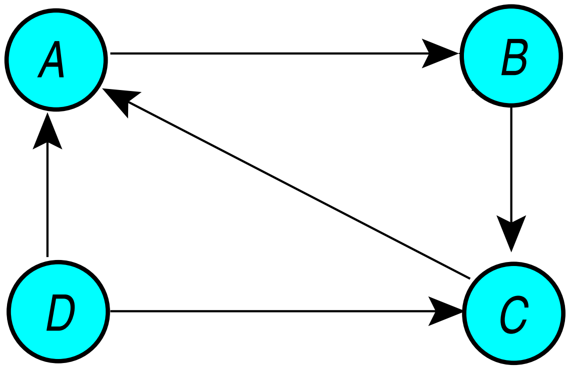


Set of possible worlds (states): $S = \{A, B, C, D\}$

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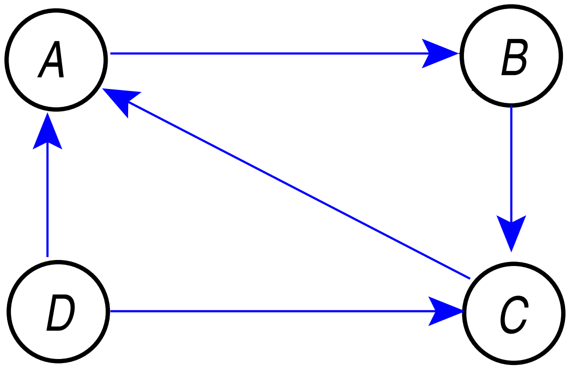
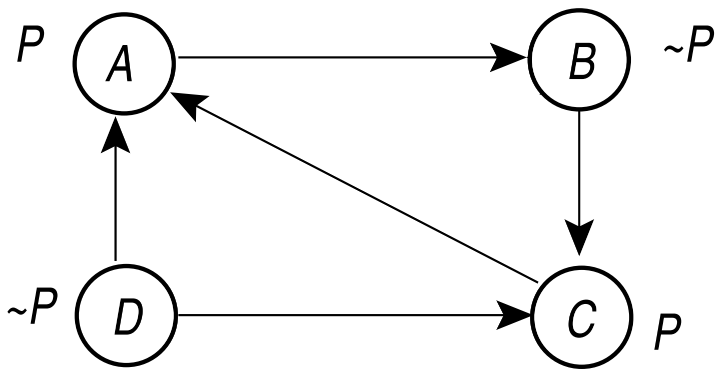



Set of possible worlds (states): $S = \{A, B, C, D\}$

Accessibility relation: $R = \{(A, B), (B, C), (C, A), (D, A), (D, C)\}$

# Example of Kripke structure




Set of possible worlds (states): $S = \{A, B, C, D\}$

Accessibility relation: $R = \{(A, B), (B, C), (C, A), (D, A), (D, C)\}$

Interpretation: $\mathcal{I} : \Pi \times S \to \{0, 1\}$

$\mathcal{I}(P, A) = 1, \mathcal{I}(P, B) = 0, \mathcal{I}(P, C) = 1, \mathcal{I}(P, D) = 0$

**Notation** Instead of $(A, B) \in R$ we will sometimes write $ARB$.

# Notation

$K = (S, R, I)$

Instead of writing $(s, t) \in R$ we will sometimes write $sRt$.

# Modal logic: Semantics

Given: Kripke structure $K = (S, R, I)$

**Valuation:**

$val_K(p)(s) = I(p, s)$ for $p \in \Pi$

$val_K$ defined for propositional operators in the same way as in classical logic

$$val_K(\Box A)(s) = \begin{cases} 1 & \text{if } val_K(A)(s') = 1 \text{ for all } s' \in S \text{ with } sRs' \\ 0 & \text{otherwise} \end{cases}$$

$$val_K(\Diamond A)(s) = \begin{cases} 1 & \text{if } val_K(A)(s') = 1 \text{ for at least one } s' \in S \text{ with } sRs' \\ 0 & \text{otherwise} \end{cases}$$

# Models, Validity, and Satisfiability

$\mathcal{F} = (S, R)$, $\mathcal{K} = (S, R, I)$

$F$ is true in $\mathcal{K}$ at a world $s \in S$:

$$(\mathcal{K}, s) \models F :\Leftrightarrow \text{val}_{\mathcal{K}}(F)(s) = 1$$

$F$ is true in $\mathcal{K}$

$$\mathcal{K} \models F :\Leftrightarrow (\mathcal{K}, s) \models F \text{ for all } s \in S$$

$F$ is true in the frame $\mathcal{F} = (S, R)$

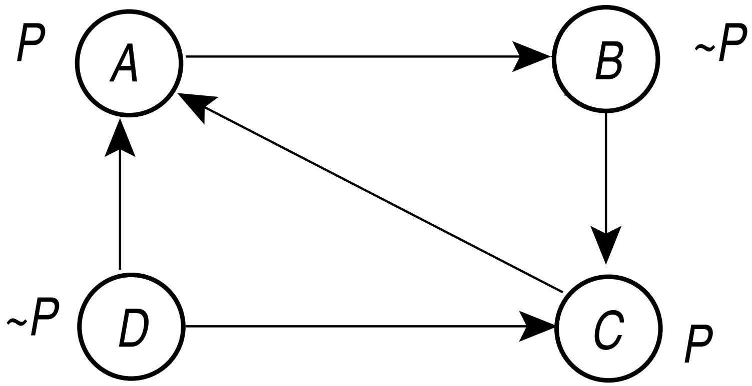
$$\mathcal{F} \models F :\Leftrightarrow (\mathcal{K}_{\mathcal{F}}) \models F \text{ for all Kripke structures } \mathcal{K}_{\mathcal{F}} = (S, R, I')$$

$$\text{defined on frame } \mathcal{F}$$

If $\Phi$ is a class of frames, $F$ is true (valid) in $\Phi$

$$\Phi \models F :\Leftrightarrow \mathcal{F} \models F \text{ for all } \mathcal{F} \in \Phi.$$

# Example for evaluation




| | | | |
|---|---|---|---|
| $(\mathcal{K}, A) \models P$ | $(\mathcal{K}, B) \models \neg P$ | $(\mathcal{K}, C) \models P$ | $(\mathcal{K}, D) \models \neg P$ |
| $(\mathcal{K}, A) \models \Box\neg P$ | $(\mathcal{K}, B) \models \Box P$ | $(\mathcal{K}, C) \models \Box P$ | $(\mathcal{K}, D) \models \Box P$ |
| $(\mathcal{K}, A) \models \Box\Box P$ | $(\mathcal{K}, B) \models \Box\Box P$ | $(\mathcal{K}, C) \models \Box\Box\neg P$ | ... |

# Entailment and Equivalence

**In classical logic we proved:**

**Proposition:**
$F$ entails $G$ iff $(F \to G)$ is valid

Does such a result hold in modal logic?

# Entailment

**In classical logic we proved:**

**Proposition:**
$F \models G$ iff $(F \rightarrow G)$ is valid

Does such a result hold in modal logic?

Need to define what $F \models G$ means

# Entailment

**Goal:** definition for $N \models F$, where $N$ is a family of modal formulae

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**Tentative 1:**

$N \models_G F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$:

$$\text{If } \mathcal{K} \models G \text{ for every } G \in N \text{ then } \mathcal{K} \models F$$

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"global entailment"

# Example

$N \models_G F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$:

$$\text{If } \mathcal{K} \models G \text{ for every } G \in N \text{ then } \mathcal{K} \models F$$

**Task:** Show that $P \models_G \Box P$

Proof: Let $\mathcal{K} = (S, R, I)$ be a Kripke structure.

Assume that $\mathcal{K} \models P$, i.e. for every $s \in S$ we have $(\mathcal{K}, s) \models P$.

Then it follows that for every $s \in S$ we have $(\mathcal{K}, s) \models \Box P$.

By the definition of $\models_G$ it follows that $P \models_G \Box P$.

# Example

$N \models_G F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$:

$$\text{If } \mathcal{K} \models G \text{ for every } G \in N \text{ then } \mathcal{K} \models F$$

**Proved:** $P \models_G \Box P$

**Question:** Is it true that $P \rightarrow \Box P$ is true in all Kripke structures?

Answer: Let $\mathcal{K} = (S, R, I)$, where

$S = \{s_1, s_2\}$, $R = \{(s_1, s_2)\}$, $I(P, s_1) = 1, I(P, s_2) = 0$.

Then $(\mathcal{K}, s_1) \models P$, $(\mathcal{K}, s_1) \not\models \Box p$.

Hence $(\mathcal{K}, s_1) \not\models P \rightarrow \Box P$.

# Entailment

**Goal:** definition for $N \models F$, where $N$ is a family of modal formulae

**Tentative 2:**

$N \models_L F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$ and every $s \in S$:

$$\text{If } (\mathcal{K}, s) \models G \text{ for every } G \in N \text{ then } (\mathcal{K}, s) \models F$$

# Entailment

**Goal:** definition for $N \models F$, where $N$ is a family of modal formulae

**Tentative 2:**

$N \models_L F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$ and every $s \in S$:

$$\text{If } (\mathcal{K}, s) \models G \text{ for every } G \in N \text{ then } (\mathcal{K}, s) \models F$$

"local entailment"

# Entailment

$N \models_G F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$:

$$\text{If } \mathcal{K} \models G \text{ for every } G \in N \text{ then } \mathcal{K} \models F$$

$N \models_L F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$ and every $s \in S$:

$$\text{If } (\mathcal{K}, s) \models G \text{ for every } G \in N \text{ then } (\mathcal{K}, s) \models F$$

**Remark:** The two entailment relations are different

$P \models_G \Box P$ (was shown before)

$P \not\models_L \Box P$

# Entailment

$N \models_G F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$:

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**Remark:** The two entailment relations are different

$P \models_G \Box P$ (was shown before)

$P \not\models_L \Box P$

Proof: Let $\mathcal{K} = (S, R, I)$, where

$S = \{s_1, s_2\}$, $R = \{(s_1, s_2)\}$, $I(P, s_1) = 1, I(P, s_2) = 0$.

Then $(\mathcal{K}, s_1) \models P$, but $(\mathcal{K}, s_1) \not\models \Box P$. Hence, $P \not\models_L \Box P$.

# Entailment

**Theorem (The deduction theorem)** The following are equivalent:

(1) $F \models_L G$

(2) $\{F, \neg G\}$ is unsatisfiable

(3) $\models (F \rightarrow G)$

(4) $\models_L (F \rightarrow G)$

Proof. $F \models_L G$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$ and every $s \in S$:
If $(\mathcal{K}, s) \models F$ then $(\mathcal{K}, s) \models G$
iff there is no Kripke structure $\mathcal{K} = (S, R, I)$ and no $s \in S$ with $(\mathcal{K}, s) \models F \wedge \neg G$
iff $\{F, \neg G\}$ is unsatisfiable

From propositional logic we know that $\{F, \neg G\}$ is unsatisfiable iff $F \rightarrow G$ is valid. This happens iff $\models_L F \rightarrow G$

# Modal Logic: Valid Formulae

Valid:

- $\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$
- $(\Box P \wedge \Box(P \rightarrow Q)) \rightarrow \Box Q$
- $(\Box P \vee \Box Q) \rightarrow \Box(P \vee Q)$
- $(\Box P \wedge \Box Q) \leftrightarrow \Box(P \wedge Q)$
- $\Box P \leftrightarrow \neg\Diamond\neg P$
- $\Diamond(P \vee Q) \leftrightarrow (\Diamond P \vee \Diamond Q)$
- $\Diamond(P \wedge Q) \rightarrow (\Diamond P \wedge \Diamond Q)$

# Modal Logic: Valid Formulae

Valid:

- $\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$
- $(\Box P \wedge \Box(P \rightarrow Q)) \rightarrow \Box Q$
- $(\Box P \vee \Box Q) \rightarrow \Box(P \vee Q)$
- $(\Box P \wedge \Box Q) \leftrightarrow \Box(P \wedge Q)$
- $\Box P \leftrightarrow \neg\Diamond\neg P$
- $\Diamond(P \vee Q) \leftrightarrow (\Diamond P \vee \Diamond Q)$
- $\Diamond(P \wedge Q) \rightarrow (\Diamond P \wedge \Diamond Q)$

Not valid:

- $\Box(P \vee Q) \rightarrow (\Box P \vee \Box Q)$
- $(\Diamond P \wedge \Diamond Q) \rightarrow \Diamond(P \wedge Q)$

# Modal Logic: Valid Formulae

Not valid: $\Box(P \vee Q) \rightarrow (\Box P \vee \Box Q)$

[explanations on the blackboard]

# Exercises

1. Show that $\diamond T$ and the schema $\Box A \rightarrow \diamond A$ have exactly the same models.

2. Exhibit a frame in which $\Box \perp$ is valid.

3. In any model $\mathcal{K}$,

   (i) if $A$ is a tautology then $\mathcal{K} \models A$;

   (ii) if $\mathcal{K} \models A$ and $\mathcal{K} \models A \rightarrow B$, then $\mathcal{K} \models B$;

   (iii) if $\mathcal{K} \models A$ then $\mathcal{K} \models \Box A$.

# Correspondence Theory

# Correspondence Theory

**Main questions:**

Assume that we consider a set of frames for which the accessibility relation has certain properties. Is it the case that in all frames in this class a certain modal formula holds?

Given a modal formula. Can we describe the frames in which the formula holds, e.g. by specifying certain properties of the accessibility relation?

# Example

Let ReflFrames be the class of all frames $\mathcal{F} = (S, R)$ in which $R$ is reflexive.

**Theorem.** For every formula $A$, the formula $\Box A \rightarrow A$ is true in all frames $\mathcal{F} = (S, R) \in$ ReflFrames (i.e. in all frames $\mathcal{F} = (S, R)$ with $R$ reflexive).

[Proof on the blackboard]

# Example

Let ReflFrames be the class of all frames $\mathcal{F} = (S, R)$ in which $R$ is reflexive.

**Theorem.** For every formula $A$, the formula $\Box A \rightarrow A$ is true in all frames in ReflFrames.

**Theorem.** If the formula $\Box A \rightarrow A$ is true in a frame $\mathcal{F} = (S, R)$ for every formula $A$, then $R$ must be reflexive.

[Proof on the blackboard]

# Conditions on $R$

The following is a list of properties of a binary relation R that are denned by first-order sentences.

1. Reflexive: $\forall s\ (sRs)$
2. Symmetric: $\forall s \forall t\ (sRt \rightarrow tRs)$
3. Serial: $\forall s \exists t\ (sRt)$
4. Transitive: $\forall s \forall t \forall u\ (sRt \wedge tRu \rightarrow sRu)$
5. Euclidean: $\forall s \forall t \forall u\ (sRt \wedge sRu \rightarrow tRu)$
6. Partially functional: $\forall s \forall t \forall u\ (sRt \wedge sRu \rightarrow t = u)$
1. Functional: $\forall s \exists t (sRt)$
8. Weakly dense: $\forall s \forall t (sRt \rightarrow \exists u\ (sRu \wedge uRt))$
9. Weakly connected: $\forall s \forall t \forall u\ (sRt \wedge sRu \rightarrow tRu \vee t = u \vee uRt)$
10. Weakly directed: $\forall s \forall t \forall u\ (sRt \wedge sRu \rightarrow \exists v (tRv \wedge uRv))$

# List of schemata of modal formulae

Corresponding to the list of properties of $R$ is a list of schemata:

1. $\Box A \longrightarrow A$
2. $A \longrightarrow \Box\Diamond A$
3. $\Box A \longrightarrow \Diamond A$
4. $\Box A \longrightarrow \Box\Box A$
5. $\Diamond A \longrightarrow \Box\Diamond A$
6. $\Diamond A \longrightarrow \Box A$
7. $\Diamond A \longleftrightarrow \Box A$
8. $\Box\Box A \longrightarrow \Box A$
9. $\Box(A \wedge \Box A \longrightarrow B) \vee \Box(B \wedge \Box B \longrightarrow A)$
10. $\Diamond\Box A \longrightarrow \Box\Diamond A$

# Correspondence theorems

| **Properties of** $R$ | | **Axioms** |
|---|---|---|
| 1. Reflexive: | $\forall s\ (sRs)$ | $\Box A \longrightarrow A$ |
| 2. Symmetric: | $\forall s \forall t\ (sRt \longrightarrow tRs)$ | $A \longrightarrow \Box \Diamond A$ |
| 3. Serial: | $\forall s \exists t\ (sRt)$ | $\Box A \longrightarrow \Diamond A$ |
| 4. Transitive: | $\forall s \forall t \forall u\ (sRt \wedge tRu \longrightarrow sRu)$ | $\Box A \longrightarrow \Box \Box A$ |
| 5. Euclidean: | $\forall s \forall t \forall u\ (sRt \wedge sRu \longrightarrow tRu)$ | $\Diamond A \longrightarrow \Box \Diamond A$ |
| 6. Partially functional: | $\forall s \forall t \forall u\ (sRt \wedge sRu \longrightarrow t = u)$ | $\Diamond A \longrightarrow \Box A$ |
| 7. Functional: | $\forall s \exists t (sRt)$ | $\Diamond A \longleftrightarrow \Box A$ |
| 8. Weakly dense: | $\forall s \forall t (sRt \longrightarrow \exists u\ (sRu \wedge uRt))$ | $\Box \Box A \longrightarrow \Box A$ |
| 9. Weakly connected: | $\forall s \forall t \forall u\ (sRt \wedge sRu \longrightarrow tRu \vee t = u \vee uRt)$ | $\Box (A \wedge \Box A \longrightarrow B) \vee \Box (B \wedge \Box B \longrightarrow A)$ |
| 10. Weakly directed: | $\forall s \forall t \forall u\ (sRt \wedge sRu \longrightarrow \exists v (tRv \wedge uRv))$ | $\Diamond \Box A \longrightarrow \Box \Diamond A$ |

**Theorem.** Let $\mathcal{F} = (S, R)$ be a frame.

Then for each of the properties 1-10, if $R$ satisfies the property, then the corresponding schema is valid in $\mathcal{F}$.

**Theorem.** If a frame $\mathcal{F} = (S, R)$ validates any one of the schemata 1-10, then $R$ satisfies the corresponding property.

# Correspondence theorems

**Theorem.** Let $\mathcal{F} = (S, R)$ be a frame.

Then for each of the properties 1-10, if $R$ satisfies the property, then the corresponding schema is valid in $\mathcal{F}$.

Proof. We illustrate with the case of transitivity. Suppose that $R$ is transitive. Let $\mathcal{K}$ be any model on $\mathcal{F}$.

To show that $\mathcal{K} \models \Box A \rightarrow \Box\Box A$, take any $s \in S$ with $(\mathcal{K}, s) \models \Box A$.

We have to prove that $(\mathcal{K}, s) \models \Box\Box A$, i.e. we have to show that $sRt$ implies $(\mathcal{K}, t) \models \Box A$, or, in other words,

$$sRt \text{ implies } (tRu \text{ implies } (\mathcal{K}, u) \models A.$$

Suppose $sRt$. If $tRu$, we have $sRu$ by transitivity, so $(\mathcal{K}, u) \models A$ since $(\mathcal{K}, s) \models \Box A$ by hypothesis.

The other cases are left as exercises.

# Correspondence theorems

**Theorem.** If a frame $\mathcal{F} = (S, R)$ validates any one of the schemata 1-10, then $R$ satisfies the corresponding property.

Proof. Consider schema 10. To show that $R$ is weakly directed, suppose $sRt$ and $sRu$.

Let $\mathcal{K}$ be any model on $\mathcal{F}$ in which $I(p)(v) = 1$ iff $uRv$.

Then by definition, $uRv$ implies $(\mathcal{K}, v) \models p$, so $(\mathcal{K}, u) \models \Box p$, and hence, as $sRu$, $(\mathcal{K}, s) \models \Diamond\Box p$. But then as schema 10 is valid in $\mathcal{F}$, $(\mathcal{K}, s) \models \Box\Diamond p$, so as $sRt$, $(\mathcal{K}, t) \models \Diamond p$. This implies that there exists a $v$ with $tRv$ and $(\mathcal{K}, v) \models p$, i.e. $V(p)(v) = 1$, so $uRv$; as desired.

# Correspondence theorems

**Theorem.** If a frame $\mathcal{F} = (S, R)$ validates any one of the schemata 1-10, then $R$ satisfies the corresponding property.

Proof. Consider now schema 8. Suppose $sRt$. Let $\mathcal{K}$ be a Kripke model on $\mathcal{F}$ with $I(p)(v) = 1$ iff $t \neq v$.

Then $(\mathcal{K}, t) \not\models p$, so $(\mathcal{K}, s) \not\models \Box p$.

Hence by validity of schema 8, $(\mathcal{K}, s) \not\models \Box\Box p$, so there exists a $u$ with $sRu$ and $(\mathcal{K}, u) \not\models \Box p$.

Then for some $v$, $uRv$ and $(\mathcal{K}, v) \not\models p$, i.e. $v = t$, so that $uRt$, as needed to show that $R$ is weakly dense.