Non-classical logics

Lecture 13: Modal logics (Part 3)

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Modal logic

Syntax

Semantics

Kripke models

global and local entailment; deduction theorem

Correspondence theory

Correspondence Theory

Main questions:

Assume that we consider a set of frames for which the accessibility relation has certain properties. Is it the case that in all frames in this class a certain modal formula holds?

Given a modal formula. Can we describe the frames in which the formula holds, e.g. by specifying certain properties of the accessibility relation?

Correspondence theorems

Properties of R		**Axioms**
1. Reflexive:	$\forall s\ (sRs)$	$\Box A \longrightarrow A$
2. Symmetric:	$\forall s \forall t\ (sRt \longrightarrow tRs)$	$A \longrightarrow \Box\Diamond A$
3. Serial:	$\forall s \exists t\ (sRt)$	$\Box A \longrightarrow \Diamond A$
4. Transitive:	$\forall s \forall t \forall u\ (sRt \wedge tRu \longrightarrow sRu)$	$\Box A \longrightarrow \Box\Box A$
5. Euclidean:	$\forall s \forall t \forall u\ (sRt \wedge sRu \longrightarrow tRu)$	$\Diamond A \longrightarrow \Box\Diamond A$
6. Partially functional:	$\forall s \forall t \forall u\ (sRt \wedge sRu \longrightarrow t = u)$	$\Diamond A \longrightarrow \Box A$
7. Functional:	$\forall s \exists t (sRt)$	$\Diamond A \longleftrightarrow \Box A$
8. Weakly dense:	$\forall s \forall t (sRt \longrightarrow \exists u\ (sRu \wedge uRt))$	$\Box\Box A \longrightarrow \Box A$
9. Weakly connected:	$\forall s \forall t \forall u\ (sRt \wedge sRu \longrightarrow tRu \vee t = u \vee uRt)$	$\Box(A \wedge \Box A \longrightarrow B) \vee \Box(B \wedge \Box B \longrightarrow A)$
10. Weakly directed:	$\forall s \forall t \forall u\ (sRt \wedge sRu \longrightarrow \exists v (tRv \wedge uRv))$	$\Diamond\Box A \longrightarrow \Box\Diamond A$

Theorem. Let $\mathcal{F} = (S, R)$ be a frame.

Then for each of the properties 1-10, if R satisfies the property, then the corresponding schema is valid in $\mathcal{F}$.

Theorem. If a frame $\mathcal{F} = (S, R)$ validates any one of the schemata 1-10, then R satisfies the corresponding property.

A general result

Property of R:

$$C(m, n, j, k): \quad \forall s_1 \forall s_2 \forall s_3 ((R^m(s_1, s_2) \wedge R^j(s_1, s_3) \longrightarrow \exists s_4 (R^n(s_2, s_4) \wedge R^k(s_3, s_4)))$$

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where $R^0(x,y) := x = y$

$$R^1(x,y) := R(x,y)$$

$$R^2(x,y) = \exists u (R(x,u) \wedge R(u,y))$$

$$R^m(x,y) = \exists u_1 \ldots u_{m-1} (R(x,u_1) \wedge \cdots \wedge R(u_{m-1},y))$$

A general result

Theorem. For every $m, n, j, k \in \mathbb{N}$, the axiom

$$\Diamond^m \Box^n P \to \Box^j \Diamond^k P$$

characterizes the class of all frames in which

$$C(m,n,j,k) : \forall s_1 \forall s_2 \forall s_3 ((R^m(s_1,s_2) \wedge R^j(s_1,s_3) \to \exists s_4 (R^n(s_2,s_4) \wedge R^k(s_3,s_4)))$$

is true.

We use the abbreviations

$$\Box^n P = \underbrace{\Box \ldots \Box}_{n \text{ times}} P$$

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In particular, $\Box^0 P$ and $\Diamond^0 P$ stand for P

A general result

Theorem. For every $m, n, j, k \in \mathbb{N}$, the axiom $\Diamond^m \Box^n P \rightarrow \Box^j \Diamond^k P$ characterizes the class of all frames in which $C(m, n, j, k)$ is true, where:

$$C(m, n, j, k) : \forall s_1 \forall s_2 \forall s_3 ((R^m(s_1, s_2) \wedge R^j(s_1, s_3) \rightarrow \exists s_4 (R^n(s_2, s_4) \wedge R^k(s_3, s_4)))$$

Proof "$\Rightarrow$" Let (S, R) be s.t. for every I $(S, R, I) \models \Diamond^m \Box^n P \rightarrow \Box^j \Diamond^k P$. We show that R has property $C(m, n, j, k)$.

Let $s_1, s_2, s_3 \in S$ be such that $R^m(s_1, s_2) \wedge R^j(s_1, s_3)$.

Let I with $I(w, P) = 1$ if $R^n(s_2, w)$ and $I(w, P) = 0$ otherwise.

Then, for $\mathcal{K} = (S, R, I)$ we have $(\mathcal{K}, s_2) \models \Box^n P$, hence $(\mathcal{K}, s_1) \models \Diamond^m \Box^n P$.

Then, by assumption, $(\mathcal{K}, s_1) \models \Box^j \Diamond^k P$.
Since $R^j(s_1, s_3)$, it follows that there exists $s \in S$ such that $R^k(s_3, s)$ and $I(s, P) = 1$, hence by the definition of I, $R^n(s_2, s)$.

A general result

Theorem. For every $m, n, j, k \in \mathbb{N}$, the axiom $\Diamond^m \Box^n P \rightarrow \Box^j \Diamond^k P$ characterizes the class of all frames in which $C(m, n, j, k)$ is true, where:

$$C(m, n, j, k) : \forall s_1 \forall s_2 \forall s_3 ((R^m(s_1, s_2) \wedge R^j(s_1, s_3) \rightarrow \exists s_4 (R^n(s_2, s_4) \wedge R^k(s_3, s_4)))$$

Proof "$\Leftarrow$" Assume $R \subseteq S \times S$ has the property $C(m, n, j, k)$.
Let $\mathcal{K} = (S, R, I)$ and $s_1 \in S$. We show that $(\mathcal{K}, s_1) \models \Diamond^m \Box^n P \rightarrow \Box^j \Diamond^k P$.

Assume that $(\mathcal{K}, s_1) \models \Diamond^m \Box^n P$.
Then there exists $s_2 \in S$ such that $R^m(s_1, s_2)$ and $(\mathcal{K}, s_2) \models \Box^n P$.

We want to show that $(\mathcal{K}, s_1) \models \Box^j \Diamond^k P$. Let $s_3 \in S$ be such that $R^j(s_1, s_3)$.
Since we assumed that R has property $C(m, n, j, k)$, there exists $s_4 \in S$ such that $R^n(s_2, s_4) \wedge R^k(s_3, s_4)$.

From $R^n(s_2, s_4)$ and $(\mathcal{K}, s_2) \models \Box^n P$ we infer that $I(P, s_4) = 1$.

From this and the fact that $R^k(s_3, s_4)$ it follows that $(\mathcal{K}, s_3) \models \Diamond^k P$.
It follows therefore that $(\mathcal{K}, s_1) \models \Box^j \Diamond^k P$. QED

Exercise

(1) Complete the proofs of the correspondence theorems.

(2) Give a property of R that is necessary and sufficient for $\mathcal{F}$ to validate the schema $A \longrightarrow \Box A$. Do the same for $\Box \perp$.

First-order definability

The correspondence theorems go a long way toward explaining the great success that the relational semantics enjoyed upon its introduction by Kripke.

Frames are much easier to deal with than modal algebras, and many modal schemata were shown to have their frames characterised by simple first-order properties of R.

For a time it seemed that propositional modal logic corresponded in strength to first-order logic, but that proved not to be so. Here are a couple of illustrations.

Examples of schemata non-definable in FOL

Example 1. The schema

$$W : \Box(\Box A \to A) \to \Box A$$

is valid in frame (S, R) iff:

(i) R is transitive, and

(ii) there is no sequence $s_0, \ldots, s_n, \ldots$ in S with $s_0 R s_1 R s_2 \ldots s_n R s_{n+1} \ldots$ for all $n \geq 0$
i.e. iff R^{-1} is well-founded.

(for a proof cf. [Boolos, 1979, p.82])

It can be shown by the Compactness Theorem of first-order logic that there exists a frame satisfying (i) and (ii) that satisfies the same first-order sentences as a frame in which (ii) fails.

Hence there can be no set of first-order sentences that defines the class of frames of this schema.

Examples of schemata non-definable in FOL

Example 2. The class of frames of the so-called McKinsey schema

$$M : \Box\Diamond A \longrightarrow \Diamond\Box A$$

is not defined by any set of first-order sentences

[Goldblatt, 1975; van Benthem, 1975]

Second order definability

Propositional modal logic corresponds to a fragment of second-order logic [Thomason, 1975].

Properties not corresp. to schemata validity

There are some naturally occurring properties of a binary relation R that do not correspond to the validity of any modal schema.

One such properties is irreflexivity, i.e. $\forall s\ \neg(sRs)$.

Proof (Idea)

Assume there exists a formula F which characterizes irreflexivity.

Consider the frame (S, R) with $S = \{s_0\}$ and $R(s_0, s_0)$.

For every frame $\mathcal{F} = (S, R)$, a frame $\mathcal{F}^* = (S^*, R^*)$ can be constructed which satisfies the same modal formulae as $\mathcal{F}$, but is irreflexive.

It would then follow that $\mathcal{F}^* \models F$, but – since in $\mathcal{F}^*$ the same formulae are true as in $\mathcal{F}$ – $(S, R) \models F$ although R is not reflexive. Contradiction.

Properties not corresp. to schemata validity

In the proof we used the following result:

Lemma. For every Kripke structure $\mathcal{K} = (S, R, I)$, a structure $\mathcal{K}^* = (S^*, R^*, I^*)$ can be constructed which satisfies the same modal formulae as $\mathcal{K}$, but R is irreflexive.

Proof: For every $s \in S$ let $s^1, s^2 \notin S$ (different). We define:

$S^* = \{s^i \mid s \in S, i = 1, 2\}$; $\quad I^*(s^i, P) = I(s, P)$ for $i = 1, 2$.
$R^*(s^i, u^j)$ iff $R(s, u)$ for all i, j if $s \neq u$.
$R^*(s^i, u^j)$ iff $R(s, s)$ and $i \neq j$.
For every formula F and every $s \in S$ the following are equivalent:

(1) $(\mathcal{K}, s) \models F$
(2) $(\mathcal{K}^*, s^1) \models F$
(3) $(\mathcal{K}^*, s^2) \models F$

[Proof by simultaneous structural induction]

Thus, $\mathcal{K} \models F$ iff $\mathcal{K}^* \models F$.

Theorem proving in modal logics

- Inference system
- Tableau calculi
- Resolution

Proof Calculi/Inference systems and proofs

Inference systems Γ (proof calculi) are sets of tuples

$$(F_1, \ldots, F_n, F_{n+1}),\ \ n \geq 0,$$

called inferences or inference rules, and written

$$\frac{\overbrace{F_1 \ \ \ldots \ \ F_n}^{\text{premises}}}{\underbrace{F_{n+1}}_{\text{conclusion}}} \ .$$

Inferences with 0 premises are also called axioms.

Proofs

A proof in Γ of a formula F from a a set of formulas N (called assumptions) is a sequence $F_1, \dots, F_k$ of formulas where

(i) $F_k = F$,

(ii) for all $1 \leq i \leq k$: $F_i \in N$, or else there exists an inference $(F_{i_1}, \ \dots, \ F_{i_{n_i}}, \ F_i)$ in Γ, such that $0 \leq i_j < i$, for $1 \leq j \leq n_i$.

Provability

Provability $\vdash_\Gamma$ of F from N in Γ:

$N \vdash_\Gamma F \quad :\Leftrightarrow$ there exists a proof Γ of F from N.

The modal system K

Axioms:

- All axioms of propositional logic (e.g. $p \vee \neg p$)
- $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$ (K)

Inference rules

$$\frac{A \qquad A \rightarrow B}{B}$$ [Modus ponens]

$$\frac{A}{\Box A}$$ [G]

Some systems of modal logic

System	*Description*
T	$K + \Box A \rightarrow A$
D	$K + \Box A \rightarrow \Diamond A$
B	$T + \neg A \rightarrow \Box \neg \Box A$
$S4$	$T + \Box A \rightarrow \Box \Box A$
$S5$	$T + \neg \Box A \rightarrow \Box \neg \Box A$
$S4.2$	$S4 + \diamond \Box A \rightarrow \Box \Diamond A$
$S4.3$	$S4 + \Box(\Box(A \rightarrow B)) \vee \Box(\Box(B \rightarrow A))$
C	$K + \frac{A \rightarrow B}{\Box(A \rightarrow B)}$ instead of (G).

Soundness and Completeness

Question:

Is it true that a formula F is valid in all frames iff F is provable in the inference system for the modal logic K?

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- F provable $\Rightarrow$ F valid in all frames: soundness
- F valid in all frames $\Rightarrow$ F provable: completeness

Soundness and Completeness

Question:

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- F valid in all frames $\Rightarrow$ F provable: completeness

Do similar results hold for other logics (taking into account correspondence theory results we proved in the last lecture)?

Soundness

Theorem. If the formula F is provable in the inference system for the modal logic K then F is valid in all frames.

Proof:

(1) All axioms of the modal logic K are valid in all frames

(2) If $(\mathcal{K}, x) \models A$ and $(\mathcal{K}, x) \models A \rightarrow B$ then $(\mathcal{K}, x) \models B$

If $\mathcal{K} \models A$ and $\mathcal{K} \models A \rightarrow B$ then $\mathcal{K} \models B$

If $\mathcal{F} \models A$ and $\mathcal{F} \models A \rightarrow B$ then $\mathcal{F} \models B$

(3) If $\mathcal{K} \models A$ then $\mathcal{K} \models \Box A$

If $\mathcal{F} \models A$ then $\mathcal{F} \models \Box A$

Completeness

Theorem. If the formula F is is valid in all frames then F is provable in the inference system for the modal logic K.

Proof

Idea:

Assume that F is not provable in the inference system for the modal logic K.

We show that:

(1) $\neg F$ is consistent with the set L of all theorems of K

(2) We can construct a "canonical" Kripke structure $\mathcal{K}_L$ and a world w in this Kripke structure such that $(\mathcal{K}, w) \models \neg F$.

Contradiction!