Non-classical logics

Lecture 14: Modal logics (Part 4)

Viorica Sofronie-Stokkermans

sofronie@uni-koblenz.de

Until now

Syntax

Semantics

Kripke models

global and local entailment; deduction theorem

Correspondence theory

First-order definability

Schemata non-definable in FOL

$W: \quad \Box(\Box A \to A) \to \Box A$

$M: \quad \Box\Diamond A \to \Diamond\Box A$

Properties binary relations that do not correspond to any modal schema

Irreflexivity, i.e. $\forall s\ \neg(sRs)$.

Until now

Theorem proving in modal logics

- Inference systems
- Tableau calculi
- Resolution

Reminder: The modal system K

Axioms:

- All axioms of propositional logic (e.g. $p \vee \neg p$)
- $\Box(A \to B) \to (\Box A \to \Box B)$ (K)

Inference rules

$$\frac{A \quad A \to B}{B} \qquad \text{[Modus ponens]}$$

$$\frac{A}{\Box A} \qquad \text{[G]}$$

Reminder: Some systems of modal logic

System	*Description*
T	$K + \Box A \to A$
D	$K + \Box A \to \Diamond A$
B	$T + \neg A \to \Box\neg\Box A$
$S4$	$T + \Box A \to \Box\Box A$
$S5$	$T + \neg\Box A \to \Box\neg\Box A$
$S4.2$	$S4 + \diamond\Box A \to \Box\Diamond A$
$S4.3$	$S4 + \Box(\Box(A \to B)) \vee \Box(\Box(B \to A))$
C	$K + \frac{A \to B}{\Box(A \to B)}$ instead of (G).

Reminder: Soundness and Completeness

Question:

Is it true that a formula F is valid in all frames iff F is provable in the inference system for the modal logic K?

- F provable $\Rightarrow$ F valid in all frames: soundness
- F valid in all frames $\Rightarrow$ F provable: completeness

Do similar results hold for other logics (taking into account correspondence theory results we proved in the last lecture)?

Reminder: Soundness

Theorem. If the formula F is provable in the inference system for the modal logic K then F is valid in all frames.

Reminder: Completeness

Theorem. If the formula F is is valid in all frames then F is provable in the inference system for the modal logic K.

Proof

Idea:

Assume that F is not provable in the inference system for the modal logic K.

We show that:

(1) $\neg F$ is consistent with the set L of all theorems of K

(2) We can construct a "canonical" Kripke structure $\mathcal{K}$ and a world w in this Kripke structure such that $(\mathcal{K}, w) \models \neg F$.

Contradiction!

Consistent sets of formulae

Let L be a set of modal formulae which:

(1) contains all propositional tautologies

(2) contains axiom K

(3) is closed under modus ponens and generalization

(4) is closed under instantiation

Definition. A subset $F \subseteq L$ is called L-inconsistent iff there exist formulae $A_1, \ldots, A_n \in F$ such that

$$(\neg A_1 \vee \cdots \vee \neg A_n) \in L$$

F is called L-consistent iff it is not L-inconsistent.

Definition. A consistent set F of modal formulae is called maximal L-consistent if for every modal formula A wither $A \in F$ or $\neg A \in F$.

Consistent sets of formulae

Let L be a set of modal formulae which:
(1) contains all propositional tautologies
(2) contains axiom K
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Typically: L : the set of all theorems of the modal logic K

Notation:

$\vdash_L F$ iff $F \in L$

$\Gamma \vdash_L F$ iff there exist formulae $G_1, \dots G_n \in \Gamma$ s.t. $\vdash_L (G_1 \rightarrow (G_2 \rightarrow \dots (G_n \rightarrow F) \dots)$

Remark: Assume L is the set of all theorems of the modal logic K.
Then F provable from Γ in modal system K iff $\Gamma \vdash_L F$.

(Induction on the length of proof)

Consistent sets of formulae

Let L be as before. In what follows we assume that L is consistent.

Theorem. Let F be a maximal L-consistent set of formulae. Then:

(1) For every formula A, either $A \in F$ or $\neg A \in F$, but not both.

(2) $A \vee B \in F$ iff $A \in F$ or $B \in F$

(3) $A \wedge B \in F$ iff $A \in F$ and $B \in F$

(4) $L \subseteq F$

(5) F is closed under Modus Ponens

Proof. (1) $A \in F$ or $\neg A \in F$ by definition.

Assume $A \in F$ and $\neg A \in F$.
We know that $\neg A \vee \neg\neg A \in L$ (propositional tautology), so F is inconsistent.
Contradiction.

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Proof. (2) "$\Rightarrow$" Assume $A \vee B \in F$, but $A, B \notin F$. Then $\neg A, \neg B \in F$. As $\neg\neg A \vee \neg\neg B \vee \neg(A \vee B) \in L$ (classical tautology) it follows that F is inconsistent.

(2) "$\Leftarrow$" Assume $A \in F$ and $A \vee B \notin F$. Then $\neg(A \vee B) \in F$. Then $\neg A \vee (A \vee B) \in L$, so F is inconsistent.

Consistent sets of formulae

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Proof. (3) Analogous to (2)

Consistent sets of formulae

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Proof. (4) If $A \in L$ then $\neg A$ is inconsistent. Hence, $\neg A \notin F$, so $A \in F$.

(5) Assume $A \in F, A \to B \in F$ and $B \notin F$, i.e. $\neg B \in F$.
Then $\neg A \vee \neg(A \to B) \vee \neg\neg B$ is a propositional tautology, hence in L.
Thus, F inconsistent.

Consistent sets of formulae

Theorem. Every consistent set F of formulae is contained in a maximally consistent set of formulae.

Proof. We enumerate all modal formulae: $A_0, A_1, \ldots$ and inductively define an ascending chain of sets of formulae:

$F_0 := F$

$$F_{n+1} := \begin{cases} F_n \cup \{A_n\} & \text{if this set is consistent} \\ F_n \cup \{\neg A_n\} & \text{otherwise} \end{cases}$$

It can be proved by induction that F_n is consistent for all n.

Let $F_{\max} = \bigcup_{n \in \mathbb{N}} F_n$.
Then $F_{\max}$ is maximal consistent and contains F.

Consistent sets of formulae

Theorem. $\{F \mid \Gamma \vdash_{\mathcal{L}} F\} = \bigcap\{\Delta \mid \Gamma \subseteq \Delta\}$,
i.e. $\Gamma \vdash_{\mathcal{L}} F$ iff F belongs to every maximal consistent set that includes Γ.

Proof. The direct implication is immediate.

We prove the converse. Instead of proving $A \Rightarrow B$ we prove that $\neg B \Rightarrow \neg A$.

If $\Gamma \not\vdash_{\mathcal{L}} F$ then $\Gamma \cup \neg F$ is $\mathcal{L}$-consistent, so it is included into some maximal consistent set Δ. So there exists a maximal consistent set which contains Γ but does not contain F.

Canonical models

Goal: Assume F is not a theorem. Construct a Kripke structure $\mathcal{K}$ and a possible world w of $\mathcal{K}$ such that $(\mathcal{K}, w) \models \neg F$.

States: State of $\mathcal{K}$: maximal consistent set of formulae.

Interpretation: $\mathcal{I}(P, W) = 1$ iff $P \in W$.

Intuition: $(\mathcal{K}, W) \models F$ iff $F \in W$.

Accessibility relation:

Intuition:
$(\mathcal{K}, W) \models \Box F$ iff for all $W', ((W, W') \in R \rightarrow (\mathcal{K}, W') \models F$

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$(\mathcal{K}, W) \models \Box F$ iff for all $W', ((W, W') \in R \rightarrow (\mathcal{K}, W') \models F$
$\Box F \in W$ iff for all $W', ((W, W') \in R \rightarrow F \in W'$

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$\Box F \in W$ iff for all $W', ((W, W') \in R \rightarrow F \in W'$

$(W, W') \in R$ iff $W' \supseteq \Box^{-1}(W) = \{F \mid \Box F \in W\}$

Canonical Kripke structure

Theorem. For every maximal consistent set W and every formula F:

$\Box F \in W$ iff for all max. consistent sets $W'[(W, W') \in R$ implies $F \in W']$

Proof. "$\Rightarrow$" follows from the definition of R.

"$\Leftarrow$" Assume that for all max. consistent sets $W', (W, W') \in R$ implies $F \in W'$, i.e.

$$\{G \mid \Box G \in W\} \subseteq W' \text{ implies } F \in W'$$

Since W' is maximal consistent it then follows that

$$\{G \mid \Box G \in W\} \vdash_{\mathcal{L}} F$$

hence $\{\Box G \mid \Box G \in W\} \vdash_{\mathcal{L}} \Box F$, so $W \vdash_{\mathcal{L}} \Box F$.

Thus, as W is a maximal consistent set of formulae, $\Box F \in W$.

Canonical Kripke structure

Theorem. $(\mathcal{K}, W) \models F$ iff $F \in W$.

Proof. Induction on the structure of the formula F.

The case $F = P$ follows from the definition of $\mathcal{I}$, while the cases $F = \perp$ and $\perp$ are immediate.

The induction step for $F = \neg F_1$ is immediate; the cases $F = F_1 \text{op} F_2$, op $\in \{\vee, \wedge\}$ follow from the properties of maximal consistent sets.

For the case $F = \Box F_1$, assume inductively that the result holds for F_1.

$$
\begin{array}{lll}
(\mathcal{K}, W) \models \Box F_1 & \text{iff} & \text{for all } W' \; ((W, W') \in R \rightarrow (\mathcal{K}, W') \models F_1) \\
 & \text{iff} & \text{for all } W' \; ((W, W') \in R \rightarrow F_1 \in W') \\
 & \text{iff} & \Box F_1 \in W \qquad \text{(we used the previous theorem)}
\end{array}
$$

Completeness

Theorem. If the formula F is is valid in all frames then F is provable in the inference system for the modal logic K.

Proof. Assume F is not provable in the inference system for K. Then $L \cup \neg F$ is consistent, hence it is included in a consistenly maximal set W.

Then $\neg F \in W$, so by the previous theorem, $(\mathcal{K}, W) \models \neg F$.

This contradicts the fact that we assumed that F is valid in all Kripke structures.

Other soundness and completeness results

$T = K + \Box A \to A$.

A formula F is provable in the inference system for the modal logic T iff F is is valid in all frames (S, R) with R reflexive.

$S4 = T + \Box A \to \Box\Box A$.

A formula F is provable in the inference system for the modal logic $S4$ iff F is is valid in all frames (S, R) with R transitive.

$S5 = T + \neg\Box A \to \Box\neg\Box A$.

A formula F is provable in the inference system for the modal logic $S5$ iff F is is valid in all frames (S, R) with R is an equivalence relation.

Soundness/completeness: characteriz. classes

Theorem. Let $\mathcal{R}$ be a class of frames characterizable through the modal formulae $C_1, \dots C_n$, and let $K(\mathcal{R})$ be the class of all Kripke structures based on frames in $\mathcal{R}$.

Let S be the inference system obtained from K by adding $C_1, \dots, C_n$ as axioms.

A formula F is provable in the inference system for the modal logic S iff F is is valid in all Kripke structures $\mathcal{K} \in K(\mathcal{R})$.

Proof (Idea) It can be shown that if S is obtained from K by adding axioms $C_1, \dots, C_n$, then the canonical Kripke structure – constructed as in the case of the modal logic K – is in $K(\mathcal{R})$ (i.e. it is based on frames in $\mathcal{R}$).

Example: Let C_1 be the axiom schema $\Box A \rightarrow \Box\Box A$. Let L be the set of all theorems of $K + C_1$. Then all maximal L-consistent sets will contain all instances of this schema.

Let $(W, W') \in R$ and $(W', W'') \in R$.
Then $\Box F \in W$ implies $\Box\Box F \in W$, hence $\Box F \in W'$ (since $(W, W') \in R$)
so $F \in W''$ (as $(W', W'') \in R$). Thus, $(W, W'') \in R$, so R is transitive.

Modal logic

Theorem proving in modal logics

- Inference systems
- Tableau calculi
- Resolution

Tableau calculus

We use labelled formulae

TG standing for “Formula G is true”

FG standing for “Formula G is false”

Tableau calculus

Formula classes

α-Formulae	$T(A \wedge B)$, $F(A \vee B)$, $F(A \rightarrow B)$, $F(\neg A)$
β-Formulae	$T(A \vee B)$, $F(A \wedge B)$, $T(A \rightarrow B)$, $T(\neg A)$
ν-Formulae	$T\ \Box A$, $F\ \Diamond A$
π-Formulae	$T\ \Diamond A$, $F\ \Box A$

Tableau calculus

Successor formulae

α	α_1	α_2
$T(A \wedge B)$	TA	TB
$F(A \vee B)$	FA	FB
$F(A \rightarrow B)$	TA	FB
$F(\neg A)$	TA	TA

β	β_1	β_2
$T(A \vee B)$	TA	TB
$F(A \wedge B)$	FA	FB
$T(A \rightarrow B)$	TB	FA
$T(\neg A)$	FA	FA

ν	ν_0
$T\Box A$	TA
$F\Diamond A$	FA

π	π_0
$T\Diamond A$	TA
$F\Box A$	FA

Tableau calculus

Every combination of top-level operator and sign occurs in one of the above cases.

When constructing the tableau, we use signed formulae prefixed by states:

$$\sigma ZA$$

where $Z \in \{T, F\}$, A is a formula, and σ is a finite sequence of natural numbers.

For the modal logic K, σ_1 is accessible from σ iff

$$\sigma_1 = \sigma n \text{ for some natural number } n.$$