

Exercises for “Non-Classical Logics” Exercise sheet 7

Exercise 7.1: (3 P)

Use the method presented in the lecture for checking whether the formula

$$F := (P \Rightarrow Q) \vee (Q \Rightarrow P)$$

is a tautology in the Łukasiewicz logic $\mathcal{L}_{\mathbb{N}^1}$, with set of truth values $[0, 1]$. (For checking the satisfiability over $[0, 1]$ of the constraints you obtain use any method you like (e.g. make case distinctions).)

Exercise 7.2: (3 P)

Use the method presented in the lecture for showing that the formula

$$F := (P \vee \neg P)$$

is not a tautology in the Gödel logic with set of truth values $[0, 1]$. (For finding a satisfying assignment into $[0, 1]$ for the constraints you obtain – and thus a variable assignment \mathcal{A} with $\mathcal{A}(F) \neq 1$ – you can use any method you like.)

Exercise 7.3: (3 P)

Consider the following operations on $[0, 1]$:

- $\sqcup : [0, 1]^2 \rightarrow [0, 1]$ defined by $x \sqcup y = (\neg x) \Rightarrow y$;
- $\sqcap : [0, 1]^2 \rightarrow [0, 1]$ defined by $x \sqcap y = \neg((\neg x) \sqcup (\neg y))$;

where \neg, \Rightarrow are the Łukasiewicz negation resp. implication.

(1) Prove that:

- $x \sqcup y = \min(1, x + y)$
- $x \sqcap y = \max(0, x + y - 1)$

(2) Prove by induction that for every $m \in \mathbb{N}$, $m \geq 2$:

$$\sqcup^m x = \underbrace{x \sqcup x \sqcup \dots \sqcup x}_{m \text{ times}} = \min(1, m \cdot x)$$

$$\text{Infer from this that } \sqcap^m x = \underbrace{x \sqcap x \sqcap \dots \sqcap x}_{m \text{ times}} = 1 - \min(1, m \cdot (1 - x)).$$

Hint: Use the fact that $x \sqcap y = \neg((\neg x) \sqcup (\neg y))$.

(3) Use (2) to show that $\sqcup^{m-1} \neg x = 1$ if and only if $x \leq \frac{m-2}{m-1}$.

Supplementary exercises

(to be discussed in the next exercise session)

Exercise 7.4: (4 P)

With the notation in Exercise 7.3, show that:

- (1) $(x \sqcup (\sqcap^{m-2}x)) = 1$ if and only if $x \geq \frac{m-2}{m-1}$.
- (2) Let F be a propositional formula in the signature of Łukasiewicz logic, and let C_{m-1} be the following formula:

$$C_{m-1} := (\sqcup^{m-1} \neg P) \wedge (P \sqcup (\sqcap^{m-2} P))$$

where P is a propositional variable.

From (1) and 7.3(3) infer that for every valuation $\mathcal{A} : \Pi \rightarrow [0, 1]$ the following are equivalent:

- (i) $\mathcal{A}(C_{m-1}) = 1$
- (ii) $\mathcal{A}(P) = \frac{m-2}{m-1}$

Exercise 7.5: (6 P)

Let $\mathcal{L}_n, \mathcal{L}_m$ be the finitely-valued Łukasiewicz logics with truth values $W_n = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \dots, \frac{n-2}{n-1}, 1\}$ and respectively $W_m = \{0, \frac{1}{m-1}, \frac{2}{m-1}, \dots, \frac{m-2}{m-1}, 1\}$.

- (1) Assume that $W_m \not\subseteq W_n$. Show that in this case $\frac{1}{m-1} \notin W_n$ and $\frac{m-2}{m-1} \notin W_n$.
- (2) Let C_m be the formula defined in Exercise 7.4(2) and $G := \sqcup^{n-1} \neg C_{m-1}$. Show that:
 - (i) for every valuation $\mathcal{A} : \Pi \rightarrow W_m$ if $\mathcal{A}(P) = \frac{m-2}{m-1}$ then $\mathcal{A}(G) = 0$.
 - (ii) for every valuation $\mathcal{A} : \Pi \rightarrow W_n$, $\mathcal{A}(G) = 1$.
- (3) Assume that $W_m \not\subseteq W_n$, hence, by (1), that $\frac{m-2}{m-1} \notin W_n$. Show that in this case G is an \mathcal{L}_n -tautology, but is not an \mathcal{L}_m -tautology.
- (4) Assume that $\text{Tautologies}(\mathcal{L}_n) \subseteq \text{Tautologies}(\mathcal{L}_m)$.
 - (i) Show that then $W_m \subseteq W_n$.
 - (ii) Use the fact that $\frac{1}{m-1} \in W_n$ to show that $(n-1)$ is a multiple of $(m-1)$, i.e. there exists $k \in \mathbb{N}$ such that $n-1 = k \cdot (m-1)$.

Please submit your solution until Tuesday, December 10, 2013, at 16:00. Joint solutions prepared by up to three persons are allowed. Please do not forget to write your name(s) on your solution.

Submission possibilities:

- By e-mail to sofronie@uni-koblenz.de with the keyword "Homework Non-Classical Logics" in the subject.
- Put it in the box in front of Room B 222.