Non-classical logics

Lecture 6: Many-valued logics (Part 2)

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Until now

- Many-valued logic (finitely-valued; infinitely-valued)

    History and Motivation

    Syntax

    Semantics

1 Syntax

- propositional variables Π
- logical operations $\mathcal{F}$

Propositional Formulas $F_{\Pi}^{\mathcal{F}}$ is the set of propositional formulas over Π defined as follows:

$$
\begin{array}{lcll}
F, G, H & ::= & c & \text{(c constant logical operator)} \\
 & \mid & P, \quad P \in \Pi & \text{(atomic formula)} \\
 & \mid & f(F_1, \ldots, F_n) & (f \in \mathcal{F} \text{ with arity } n)
\end{array}
$$

Semantics

We assume that a set $M = \{w_1, w_2, \dots, w_m\}$ of truth values is given.
We assume that a subset $D \subseteq M$ of designated truth values is given.

1. Meaning of the logical operators

$f \in \mathcal{F}$ with arity $n \quad \mapsto \quad f_M : M^n \to M$ (truth tables for the operations in $\mathcal{F}$)

2. The meaning of the propositional variables

A Π-valuation is a map $\mathcal{A} : \Pi \to M$.

3. Truth value of a formula in a valuation

Given an interpretation of the operation symbols $(M, \{f_M\}_{f \in \mathcal{F}})$, any Π-valuation $\mathcal{A} : \Pi \to M$, can be extended to $\mathcal{A}^* : \Sigma$-formulas $\to M$.

$$\begin{aligned} \mathcal{A}^*(c) &= c_M (\text{for every constant operator } c \in \mathcal{F}) \\ \mathcal{A}^*(P) &= \mathcal{A}(P) \\ \mathcal{A}^*(f(F_1, \dots, F_n)) &= f_M(\mathcal{A}^*(F_1), \dots, \mathcal{A}^*(F_n)) \end{aligned}$$

For simplicity, we write $\mathcal{A}$ instead of $\mathcal{A}^*$.

Models, Validity, and Satisfiability

$M = \{w_1, \ldots, w_m\}$ set of truth values
$D \subseteq M$ set of designated truth values
$\mathcal{A} : \Pi \to M$.

F is valid in $\mathcal{A}$ ($\mathcal{A}$ is a model of F; F holds under $\mathcal{A}$):

$$\mathcal{A} \models F :\Leftrightarrow \mathcal{A}(F) \in D$$

F is valid (or is a tautology):

$$\models F :\Leftrightarrow \mathcal{A} \models F \text{ for all } \Pi\text{-valuations } \mathcal{A}$$

F is called satisfiable iff there exists an $\mathcal{A}$ such that $\mathcal{A} \models F$.

Otherwise F is called unsatisfiable (or contradictory).

The logic $\mathcal{L}_3$

Set of truth values: $M = \{1, u, 0\}$.

Designated truth values: $D = \{1\}$.

Logical operators: $\mathcal{F} = \{\vee, \wedge, \neg, \sim\}$.

Truth tables for the operators

$\vee$	0	u	1
0	0	u	1
u	u	u	1
1	1	1	1

$\wedge$	0	u	1
0	0	0	0
u	0	u	u
1	0	u	1

$v(F \wedge G) = \min(v(F), v(G))$
$v(F \vee G) = \max(v(F), v(G))$

Under the assumption that $0 < u < 1$.

Truth tables for negations

A	$\neg A$	$\sim A$	$\sim \neg A$	$\sim\sim A$	$\neg\neg A$	$\neg \sim A$
1	0	0	1	1	1	1
u	u	1	1	0	u	0
0	1	1	0	0	0	0

Translation in natural language:

$v(A) = 1$ gdw. A is true

$v(\neg A) = 1$ gdw. A is false

$v(\sim A) = 1$ gdw. A is not true

$v(\sim \neg A) = 1$ gdw. A is not false

First-order many-valued logic

1. Syntax

- non-logical symbols (domain-specific)
  $\Rightarrow$ terms, atomic formulas

- logical symbols $\mathcal{F}$, quantifiers
  $\Rightarrow$ formulae

Signature;Variables; Terms/Atoms/Formulae

Signature: $\Sigma = (\Omega, \Pi)$, where

- Ω: set of function symbols f with arity $n \geq 0$, written f/n,
- Π: set of predicate symbols p with arity $m \geq 0$, written p/m.

Variables: Countably infinite set X.

Terms: As in classical logic

Atoms: (atomic formulas) over Σ are formed according to this syntax:

$$A, B \quad ::= \quad p(s_1, ..., s_m) \quad , p/m \in \Pi$$

Formulae:

$\mathcal{F}$ set of logical operations; $\mathcal{Q} = \{Q_1, \ldots, Q_k\}$ set of quantifiers

$F_\Sigma(X)$ is the set of first-order formulas over Σ defined as follows:

$$\begin{array}{llll} F, G, H & ::= & c & (c \in \mathcal{F}, \text{ constant}) \\ & \mid & A & (\text{atomic formula}) \\ & \mid & f(F_1, \ldots, F_n) & (f \in \mathcal{F} \text{ with arity } n) \\ & \mid & QxF & (Q \in \mathcal{Q} \text{ is a quantifier}) \end{array}$$

Semantics

- **Truth values; Interpretation of logical symbols** $M = \{1, \dots, m\}$ set of truth values; $D \subseteq M$ set of designated truth values.
  – Truth tables for the logical operations: $\{f_M : M^n \rightarrow M | f/n \in \mathcal{F}\}$
  – "Truth tables" for the quantifiers: $\{Q_M : \mathcal{P}(M) \rightarrow M | Q \in \mathcal{Q}\}$

- **Interpretation of non-logical variables:** M-valued Σ-structure
  $\mathcal{A} = (U, \ (f_{\mathcal{A}} : U^n \rightarrow U)_{f/n \in \Omega}, \ (p_{\mathcal{A}} : U^m \rightarrow M)_{p/m \in \Pi})$
  where $U \neq \emptyset$ is a set, called the universe of $\mathcal{A}$.

- **Variable assignments:** $\beta : X \rightarrow \mathcal{A}$ and extensions to terms $\mathcal{A}(\beta) : T_\Sigma \rightarrow \mathcal{A}$ as in classical logic.

- **Truth value of a formula in $\mathcal{A}$ with respect to β** $\mathcal{A}(\beta) : \mathsf{F}_\Sigma(X) \rightarrow M$ is defined inductively as follows:

$$
\begin{aligned}
\mathcal{A}(\beta)(c) &= c_M \\
\mathcal{A}(\beta)(p(s_1, \dots, s_n)) &= p_{\mathcal{A}}(\mathcal{A}(\beta)(s_1), \dots, \mathcal{A}(\beta)(s_n)) \in M \\
\mathcal{A}(\beta)(f(F_1, \dots, F_n)) &= f_M(\mathcal{A}(\beta)(F_1), \dots, \mathcal{A}(\beta)(F_n)) \\
\mathcal{A}(\beta)(QxF) &= Q_M(\{\mathcal{A}(\beta[x \mapsto a])(F) \mid a \in U\})
\end{aligned}
$$

First-order version of $\mathcal{L}_3$

$M = \{0, u, 1\}, \quad D = \{1\}$

$\mathcal{F} = \{\vee, \wedge, \neg, \sim\};$ truth values as the propositional version

$\mathcal{Q} = \{\forall, \exists\}$

$$\forall_M(S) = \begin{cases} 1 & \text{if } S = \{1\} \\ 0 & \text{if } 0 \in S \\ u & \text{otherwise} \end{cases} \qquad \exists_M(S) = \begin{cases} 1 & \text{if } 1 \in S \\ 0 & \text{if } S = \{0\} \\ u & \text{otherwise} \end{cases}$$

$\mathcal{A}(\beta)(\forall x F(x)) = 1$ *iff* for all $a \in U_{\mathcal{A}}$, $\mathcal{A}(\beta[x \mapsto a])(F(x)) = 1$

$\mathcal{A}(\beta)(\forall x F(x)) = 0$ *iff* for some $a \in U_{\mathcal{A}}$, $\mathcal{A}(\beta[x \mapsto a])(F(x)) = 0$

$\mathcal{A}(\beta)(\forall x F(x)) = u$ otherwise

$\mathcal{A}(\beta)(\exists x F(x)) = 1$ *iff* for some $a \in U_{\mathcal{A}}$, $\mathcal{A}(\beta[x \mapsto a])(F(x)) = 1$

$\mathcal{A}(\beta)(\exists x F(x)) = 0$ *iff* for all $a \in U_{\mathcal{A}}$, $\mathcal{A}(\beta[x \mapsto a])(F(x)) = 0$

$\mathcal{A}(\beta)(\forall x F(x)) = u$ otherwise

Models, Validity, and Satisfiability

F is valid in $\mathcal{A}$ under assignment β:

$$\mathcal{A}, \beta \models F \quad :\Leftrightarrow \quad \mathcal{A}(\beta)(F) \in D$$

F is valid in $\mathcal{A}$ ($\mathcal{A}$ is a model of F):

$$\mathcal{A} \models F \quad :\Leftrightarrow \quad \mathcal{A}, \beta \models F, \text{ for all } \beta \in X \to U_{\mathcal{A}}$$

F is valid:

$$\models F \quad :\Leftrightarrow \quad \mathcal{A} \models F, \text{ for all } \mathcal{A} \in \Sigma\text{-alg}$$

F is called satisfiable iff there exist $\mathcal{A}$ and β such that $\mathcal{A}, \beta \models F$.
Otherwise F is called unsatisfiable.

Entailment

$N \models F$ $:\Leftrightarrow$ for all $\mathcal{A} \in \Sigma$-alg and $\beta \in X \to U_{\mathcal{A}}$:
if $\mathcal{A}(\beta)(G) \in D$, for all $G \in N$, then $\mathcal{A}(\beta)(F) \in D$.

Entailment

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Goal: Define a version of implication '$\Rightarrow$' such that

$F \models G$ iff $\models F \Rightarrow G$

Weak implication

The logical operations $\supset$ and $\equiv$ are introduced as defined operations:

Weak implication

$$F \supset G :=\sim F \vee G$$

Weak equivalence

$$F \equiv G := (F \supset G) \wedge (G \supset F)$$

$F \supset G$	1	u	0
1	1	u	0
u	1	1	1
0	1	1	1

$F \equiv G$	1	u	0
1	1	u	0
u	u	1	1
0	0	1	1

Strong implication

The logical operations $\rightarrow$ and $\leftrightarrow$ are introduced as defined operations:

Strong implication

$$F \rightarrow G := \neg F \vee G$$

Strong equivalence

$$F \leftrightarrow G := (F \rightarrow G) \wedge (G \rightarrow F)$$

$F \rightarrow G$	1	u	0
1	1	u	0
u	1	u	u
0	1	1	1

$F \leftrightarrow G$	1	u	0
1	1	u	0
u	u	u	u
0	0	u	1

Comparisons

Implications

$A \supset B$	1	u	0
1	1	u	0
u	1	1	1
0	1	1	1

$A \to B$	1	u	0
1	1	u	0
u	1	u	u
0	1	1	1

Equivalences

$A \equiv B$	1	u	0
1	1	u	0
u	u	1	1
0	0	1	1

$A \leftrightarrow B$	1	u	0
1	1	u	0
u	u	u	u
0	0	u	1

Equivalences

$A \supset B := \ \sim A \vee B \qquad\qquad A \to B := \neg A \vee B$

$A \equiv B := (A \supset B) \wedge (B \supset A) \qquad\qquad A \leftrightarrow B := (A \to B) \wedge (B \to A)$

$A \approx B := (A \equiv B) \wedge (\neg A \equiv \neg B) \qquad\qquad A \Leftrightarrow B := (A \leftrightarrow B) \wedge (\neg A \leftrightarrow \neg B)$

$A \text{ id } B := \ \sim\sim (A \approx B)$

A	B	$A \equiv B$	$A \leftrightarrow B$	$A \approx B$	$A \Leftrightarrow B$	$A \text{ id } B$
1	1	1	1	1	1	1
1	u	u	u	u	u	0
1	0	0	0	0	0	0
u	1	u	u	u	u	0
u	u	1	u	1	u	1
u	0	1	u	u	u	0
0	1	0	0	0	0	0
0	u	1	u	u	u	0
0	0	1	1	1	1	1

Some $\mathcal{L}_3$ tautologies

$\neg\neg A$ id A

$\sim\sim A \equiv A$

$\neg \sim A \equiv A$

$(A \wedge B) \vee C$ id $(A \vee C) \wedge (B \vee C)$

$(A \vee B) \wedge C$ id $(A \wedge C) \vee (B \wedge C)$

$\neg(A \vee B)$ id $\neg A \wedge \neg B$

$\neg(A \wedge B)$ id $\neg A \vee \neg B$

$\sim (A \vee B)$ id $\sim A \wedge \sim B$

$\sim (A \wedge B)$ id $\sim A \vee \sim B$

$\neg(\forall x A)$ id $\exists x \neg A$

$\neg(\exists x A)$ id $\forall x \neg A$

$\sim (\forall x A)$ id $\exists x \sim A$

$\sim (\exists x A)$ id $\forall x \sim A$

No occurrence of $\neg$

Lemma. Let F be a formula which does not contain the strong negation $\neg$. Then the following are equivalent:

(1) F is an $\mathcal{L}_3$-tautology.

(2) F is a two-valued tautology (negation is identified with $\sim$)

Proof.

"$\Rightarrow$" Every $\mathcal{L}_3$-tautology is a 2-valued tautology (the restriction of the operators $\vee, \wedge, \sim$ to $\{0, 1\}$ coincides with the Boolean operations $\vee, \wedge, \neg$).

"$\Leftarrow$" Assume that F is a two-valued tautology. Let $\mathcal{A}$ be an $\mathcal{L}_3$-structure and $\beta : X \rightarrow \mathcal{A}$ be a valuation. We construct a two-valued structure $\mathcal{A}'$ from $\mathcal{A}$, which agrees with $\mathcal{A}$ except for the fact that whenever $p_{\mathcal{A}}(\overline{x}) = u$ we define $p_{\mathcal{A}'}(\overline{x}) = 0$. Then $\mathcal{A}'(\beta)(F) = 1$. It can be proved that

$\mathcal{A}(\beta)(F) = 1 \Rightarrow \mathcal{A}'(\beta)(F) = 1$

$\mathcal{A}(\beta)(F) \in \{0, u\} \Rightarrow \mathcal{A}'(\beta)(F) = 0.$

Hence, $\mathcal{A}(\beta)(F) = 1$.

Exercises

1. **Let F be a formula which does not contain $\sim$.**
   Then F is not a tautology.

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   Proof. Take the valuation which maps all propositional variables to u.

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   Then F is not a tautology.

   Proof. Take the valuation which maps all propositional variables to u.

2. **Prove that for every term t, $\forall x q(x) \supset q(x)[t/x]$ is an $\mathcal{L}_3$-tautology.**

3. **Show that $\forall x q(x) \to q(x)[t/x]$ is not a tautology.**

Exercises

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3. **Show that $\forall x q(x) \to q(x)[t/x]$ is not a tautology.**

   Solution. $q \to q$ is not a tautology.

Exercises

4. **Which of the following statements are true?**

   If $F \equiv G$ is a tautology and F is a tautology then G is a tautology.

   If $F \equiv G$ is a tautology and F is satisfiable then G is satisfiable.

   If $F \equiv G$ is a tautology and F is a non-tautology then G is a non-tautology.

   If $F \equiv G$ is a tautology and F is two-valued then G is two-valued.

F is a non-tautology iff for every 3-valued structure, $\mathcal{A}$ and every valuation β, $\mathcal{A}(\beta)(F) \neq 1$.

F is two-valued iff for every 3-valued structure, $\mathcal{A}$ and every valuation β, $\mathcal{A}(\beta)(F) \in \{0, 1\}$.

Exercises

4. **Which of the following statements are true?**

   If $F \equiv G$ is a tautology and F is a tautology then G is a tautology.

   true

   If $F \equiv G$ is a tautology and F is satisfiable then G is satisfiable.

   If $F \equiv G$ is a tautology and F is a non-tautology then G is a non-tautology.

   If $F \equiv G$ is a tautology and F is two-valued then G is two-valued.

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   true

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   true

   If $F \equiv G$ is a tautology and F is a non-tautology then G is a non-tautology.

   If $F \equiv G$ is a tautology and F is two-valued then G is two-valued.

F is a non-tautology iff for every 3-valued structure, $\mathcal{A}$ and every valuation β, $\mathcal{A}(\beta)(F) \neq 1$.

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   true

   If $F \equiv G$ is a tautology and F is satisfiable then G is satisfiable.

   true

   If $F \equiv G$ is a tautology and F is a non-tautology then G is a non-tautology.

   true

   If $F \equiv G$ is a tautology and F is two-valued then G is two-valued.

F is a non-tautology iff for every 3-valued structure, $\mathcal{A}$ and every valuation $\beta, \mathcal{A}(\beta)(F) \neq 1$.

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Exercises

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   true

   If $F \equiv G$ is a tautology and F is satisfiable then G is satisfiable.

   true

   If $F \equiv G$ is a tautology and F is a non-tautology then G is a non-tautology.

   true

   If $F \equiv G$ is a tautology and F is two-valued then G is two-valued.

   false

F is a non-tautology iff for every 3-valued structure, $\mathcal{A}$ and every valuation β, $\mathcal{A}(\beta)(F) \neq 1$.

F is two-valued iff for every 3-valued structure, $\mathcal{A}$ and every valuation β, $\mathcal{A}(\beta)(F) \in \{0, 1\}$.

Functional completeness

Definition A family $(M, \{f_M : M^n \to M\}_{f \in \mathcal{F}})$ is called functionally complete if every function $g : M^m \to M$ can be expressed in terms of the functions $\{f_M : M^n \to M \mid f \in \mathcal{F}\}$.

Definition A many-valued logic with finite set of truth values M and logical operators $\mathcal{F}$ is called functionally complete if for every function $g : M^m \to M$ there exists a propositional formula F of the logic such that for every $\mathcal{A} : \Pi \to M$

$g(\mathcal{A}(x_1), \dots, \mathcal{A}(x_m)) = \mathcal{A}(F)$.

Example: Propositional logic

$F: \quad (P \vee Q) \wedge ((\neg P \wedge Q) \vee R)$

P	Q	R	$(P \vee Q)$	$\neg P$	$(\neg P \wedge Q)$	$((\neg P \wedge Q) \vee R)$	F
0	0	0	0	1	0	0	0
0	0	1	0	1	0	1	0
0	1	0	1	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	1	0	0	0	0
1	0	1	1	0	0	1	1
1	1	0	1	0	0	0	0
1	1	1	1	0	0	1	1

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P	Q	R	$(P \vee Q)$	$\neg P$	$(\neg P \wedge Q)$	$((\neg P \wedge Q) \vee R)$	F
0	0	0	0	1	0	0	0
0	0	1	0	1	0	1	0
0	1	0	1	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	1	0	0	0	0
1	0	1	1	0	0	1	1
1	1	0	1	0	0	0	0
1	1	1	1	0	0	1	1

Example: Propositional logic

$F: \quad (P \vee Q) \wedge ((\neg P \wedge Q) \vee R)$

P	Q	R	$(P \vee Q)$	$\neg P$	$(\neg P \wedge Q)$	$((\neg P \wedge Q) \vee R)$	F
0	0	0	0	1	0	0	0
0	0	1	0	1	0	1	0
0	1	0	1	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	1	0	0	0	0
1	0	1	1	0	0	1	1
1	1	0	1	0	0	0	0
1	1	1	1	0	0	1	1

DNF: $(\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge R)$

Functional completeness

Theorem. Propositional logic is functionally complete.

Proof. For every $g : \{0,1\}^m \to \{0,1\}$ let:

$F = \bigvee_{(a_1,\dots,a_m)\in\{0,1\}}(c_g(a_1,\dots,a_m) \wedge P_1^{a_1} \wedge \cdots \wedge P_m^{a_m})$

where $P^a = \begin{cases} P & \text{if } a = 1 \\ \neg P & \text{if } a = 0 \end{cases}$

(Then clearly $\mathcal{A}(P)^a = 1$ iff $\mathcal{A}(P) = a$, $\quad$ i.e. $1^1 = 0^0 = 1; 1^0 = 0^1 = 0$.)

It can be easily checked that for every $\mathcal{A} : \{P_1,\dots,P_m\} \to \{0,1\}$ we have:

$g(\mathcal{A}(P_1),\dots,\mathcal{A}(P_m)) = \mathcal{A}(F)$.

Functional completeness

Theorem. The logic $\mathcal{L}_3$ is not functionally complete.

Proof. If F is a formula with n propositional variables in the language of $\mathcal{L}_3$ with operators $\{\neg, \sim, \vee, \wedge\}$ then for the valuation $\mathcal{A} : \Pi = \{P_1, \ldots, P_n\} \rightarrow \{0, u, 1\}$ with $\mathcal{A}(P_i) = 1$ for all i we have: $\mathcal{A}(F) \neq u$.

Therefore: If g is a function which takes value u when the arguments are in $\{0, 1\}$ then there is no formula F such that $g(\mathcal{A}(P_1), \ldots, \mathcal{A}(P_n)) = \mathcal{A}(F)$ for all $\mathcal{A} : \Pi \rightarrow \{0, u, 1\}$.

Theorem. $\mathcal{L}_3^+$, obtained from $\mathcal{L}_3$ by adding one more constant operation u (which takes always value u) is functionally complete.

A simple criterion for functional completeness

Theorem. An m-valued logic with set of truth values $M = \{w_1, \dots, w_m\}$ and logical operations $\mathcal{F}$ with truth tables $\{f_M \mid f \in \mathcal{F}\}$ in which the functions:

- $\min(x, y)$, $\max(x, y)$,
- $J_k(x) = \begin{cases} 1 \text{ (maximal element)} & \text{if } k = x \\ 0 \text{ (minimal element)} & \text{otherwise} \end{cases}$
- all constant functions $c^n_k(x_1, \dots, x_n) = k$

can be expressed in terms of the functions $\{f_M \mid f \in \mathcal{F}\}$

is functionally complete.

Proof. Let $g : M^n \to M$.

$$g(x_1, \dots, x_n) =$$
$$\max\{\min\{c^n_{g(a_1,\dots,a_n)}, J_{a_1}(x_1), \dots, J_{a_n}(x_n)\} \mid (a_1, \dots a_n) \in M^n\}$$

Functional completeness of $\mathcal{L}_3^+$

Theorem. $\mathcal{L}_3^+$, obtained from $\mathcal{L}_3$ by adding one more constant operation u (which takes always value u) is functionally complete.

Proof

- We define $J_1, J_u, J_0 : \{0, u, 1\} \to \{0, u, 1\}$ as follows:

$J_0(x) = \sim\sim \neg x$

$J_u(x) = \sim x \wedge \sim \neg x$

$J_1(x) = \sim\sim x$

x	$J_0(x)$	$J_u(x)$	$J_1(x)$
0	1	0	0
u	0	1	0
1	0	0	1

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$$J_0(x) = \sim\sim \neg x$$

$$J_u(x) = \sim x \wedge \sim \neg x$$

$$J_1(x) = \sim\sim x$$

x	$J_0(x)$	$J_u(x)$	$J_1(x)$
0	1	0	0
u	0	1	0
1	0	0	1

- min and max are $\wedge$ resp. $\vee$.

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$J_u(x) = \sim x \wedge \sim \neg x$

$J_1(x) = \sim\sim x$

x	$J_0(x)$	$J_u(x)$	$J_1(x)$
0	1	0	0
u	0	1	0
1	0	0	1

- min and max are $\wedge$ resp. $\vee$.
- The constant operation u is in the language.
- The constant functions 0 and 1 are definable as follows:

$1(x) = \sim x \vee \neg \sim x$
$0(x) = \sim (\sim x \vee \neg \sim x)$

Example

Let g the following binary function:

g	0	u	1
0	0	u	0
u	u	u	u
1	0	u	0

$$
\begin{aligned}
g(x_1, x_2) = \quad & (u \wedge J_0(x_1) \wedge J_u(x_2)) \vee (u \wedge J_u(x_1) \wedge J_0(x_2)) \vee \\
& (u \wedge J_u(x_1) \wedge J_u(x_2)) \vee (u \wedge J_u(x_1) \wedge J_0(x_2)) \vee (u \wedge J_1(x_1) \wedge J_u(x_2)) \\
= \quad & (u \wedge \sim\sim \neg x_1 \wedge \sim x_2 \wedge \sim \neg x_2) \vee (u \wedge \sim x_1 \wedge \sim \neg x_1 \wedge \sim\sim \neg x_2) \vee \\
& (u \wedge \sim x_1 \wedge \sim \neg x_1 \wedge \sim x_2 \wedge \sim \neg x_2) \vee \ldots.
\end{aligned}
$$

Post logics

$P_m = \{0, 1, \ldots, m-1\}$

$\mathcal{F} = \{\vee, s\}$

$\vee_P(a, b) = \max(a, b)$

$s_P(a) = a - 1 \pmod m$

Post logics

Theorem. The Post logics are functionally complete.

Proof:

1. max is $\vee_P$
2. The functions $x - k \pmod{m}$ and $x + k \pmod{m}$ are definable

   $x - k = \underbrace{s(s(\ldots s}_{k \text{ times}}(x))) \text{ (mod m)}$

   $x + k = x - (m - k) \text{ (mod m)}, \; 0 < k < m.$

   $x + 0 = x$

3. $\min(x, y) = m - 1 - \max(m - 1 - x, m - 1 - y)$

Post logics

Theorem. The Post logics are functionally complete.

Proof:

4. All constants are definable

$T(x) = max\{x, x-1, \ldots, x-m+1\}$
$T(x) = m-1$ for all x.

The other constants are definable using s iterated $1, 2, \ldots, m-1$ times.

5. $T_k(x) = \max(\max[T(x)-1, x] - m + 1, x + k) - m + 1$ has the property that $T_k(x) = \begin{cases} 0 & \text{if } x \neq m-1 \\ k & \text{if } x = m-1 \end{cases}$

Then $J_k(x) = \max(T_{J_k(0)}(x+m-1), \ldots, T_{J_k(m-2)}(x+1), T_{J_k(m-1)}(x))$.

in general, if $g(i){=}k_i$ then $g(x){=}\max(T_{k_{m-1}}(x), T_{k_{m-2}}(x+1), \ldots, T_{k_0}(x{+}(m{-}1)))$

Other many-valued logics

Łukasiewicz logics $\mathcal{L}_n$

- Set of truth values $M = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, 1\}$
- Logical operations: $\vee, \wedge, \neg, \Rightarrow$
  - $\vee_{\text{Ł}_n} = \max$
  - $\wedge_{\text{Ł}_n} = \min$
  - $\neg_{\text{Ł}_n} x = 1 - x$
  - $x \Rightarrow_{\text{Ł}_n} y = \min(1, 1 - x + y)$
- First-order version: $\mathcal{Q} = \{\forall, \exists\}$

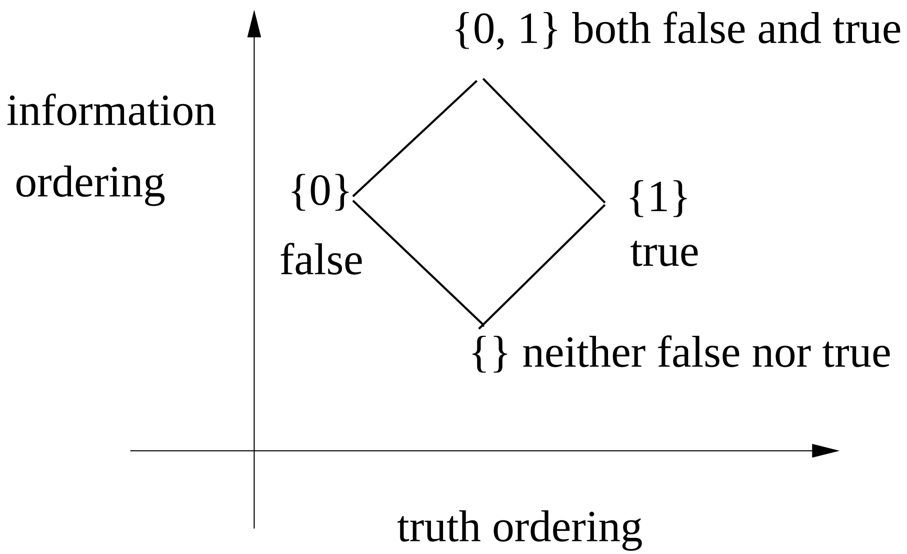
Łukasiewicz logics

Łukasiewicz implication $x \Rightarrow_{Ł_n} y = \min(1, 1 - x + y)$

$\mathcal{L}_n$

$\Rightarrow$	0	$\frac{1}{n-1}$	$\frac{2}{n-1}$	$\dots$	$\frac{n-2}{n-1}$	1
0	1	1	1	$\dots$	1	1
$\frac{1}{n-1}$	$\frac{n-2}{n-1}$	1	1	$\dots$	1	1
$\frac{2}{n-1}$	$\frac{n-3}{n-1}$	$\frac{n-2}{n-1}$	1	$\dots$	1	1
$\dots$						
1	0	$\frac{1}{n-1}$	$\frac{2}{n-1}$	$\dots$	$\frac{n-2}{n-1}$	1

Belnap's 4-valued logic



$\wedge, \vee$: sup/inf in the truth ordering

$\sim \{\} = \{\}, \quad \sim \{0,1\} = \{0,1\}, \quad \sim \{0\} = \{1\}, \quad \sim \{1\} = \{0\}$

Designated values:

Computer science: $D = \{\{1\}\}$

Other applications (e.g. information bases): $D = \{\{1\}, \{0,1\}\}$

Proof Calculi and Automated reasoning

- Axiom systems $\mapsto$ proofs
- Tableau calculi
- Resolution calculi

...

Proof Calculi/Inference systems and proofs

Inference systems Γ (proof calculi) are sets of tuples

$$(F_1, \ldots, F_n, F_{n+1}), \;\; n \geq 0,$$

called inferences or inference rules, and written

$$\frac{\overbrace{F_1 \;\; \ldots \;\; F_n}^{\text{premises}}}{\underbrace{F_{n+1}}_{\text{conclusion}}} \; .$$

Inferences with 0 premises are also called axioms.

Clausal inference system: premises and conclusions are clauses. One also considers inference systems over other data structures.

Proofs

A proof in Γ of a formula F from a a set of formulas N (called assumptions) is a sequence $F_1, \ldots, F_k$ of formulas where

(i) $F_k = F$,

(ii) for all $1 \leq i \leq k$: $F_i \in N$, or else there exists an inference $(F_{i_1}, \ \ldots, \ F_{i_{n_i}}, \ F_i)$ in Γ, such that $0 \leq i_j < i$, for $1 \leq j \leq n_i$.

Soundness and Completeness

Provability $\vdash_\Gamma$ of F from N in Γ:

$N \vdash_\Gamma F \quad :\Leftrightarrow \quad$ there exists a proof Γ of F from N.

Γ is called sound $\quad :\Leftrightarrow$

$$\frac{F_1 \quad \ldots \quad F_n}{F} \in \Gamma \quad \Rightarrow \quad F_1, \ldots, F_n \models F$$

Γ is called complete $\quad :\Leftrightarrow$

$$N \models F \quad \Rightarrow \quad N \vdash_\Gamma F$$

Γ is called refutationally complete $\quad :\Leftrightarrow$

$$N \models \bot \quad \Rightarrow \quad N \vdash_\Gamma \bot$$

Axiom systems

For $\mathcal{L}_3$: Wajsberg proposed an axiom system
(based on connectors $\neg$ and $\Rightarrow$):

$A_1 : (A \Rightarrow (B \Rightarrow A))$

$A_2 : (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$

$A_3 : (\neg A \Rightarrow \neg B) \Rightarrow (B \Rightarrow A)$

$A_4 : ((A \Rightarrow \neg A) \Rightarrow A) \Rightarrow A$

Inference rules:

Moduls Ponens: $$\frac{A \qquad A \Rightarrow B}{B}$$

Axiom systems

For $\mathcal{L}_3$: Wajsberg proposed an axiom system
(based on connectors $\neg$ and $\Rightarrow$):

$x \wedge y = x \cdot (x \Rightarrow y)$,

where $x \cdot y = \neg(x \Rightarrow \neg y)$

Proof calculi

Main disadvantage:

New proof calculus for each many-valued logic.

Goal:

Uniform methods for checking validity/satisfiability of formulae.

Automated reasoning

Classical logic:

Task: prove that F is valid

Method: prove that $\neg F$ is unsatisfiable:

– assume $\neg F$; derive a contradiction.

Automated reasoning

Classical logic:

Task: prove that F is valid

Method: prove that $\neg F$ is unsatisfiable:

– assume $\neg F$; derive a contradiction.

Many-valued logic:

Task: prove that F is valid

(i.e. $\mathcal{A}(\beta)(F) \in D$ for all $\mathcal{A}, \beta$)

Method: prove that it is not possible that $\mathcal{A}(\beta) \in M \backslash D$:

– assume $F \in M \backslash D$; derive a contradiction.

Automated reasoning

Classical logic:

Task: prove that F is valid

Method: prove that $\neg F$ is unsatisfiable:

$-$ assume $\neg F$; derive a contradiction.

Many-valued logic:

Task: prove that F is valid

(i.e. $\mathcal{A}(\beta)(F) \in D$ for all $\mathcal{A}, \beta$)

Method: prove that it is not possible that $\mathcal{A}(\beta) \in M \backslash D$:

$-$ assume $F \in M \backslash D$; derive a contradiction.

Problem: How do we express the fact that $F \in M \backslash D$

1) $\bigvee_{v \in M \backslash D} (F = v)$

2) more economical notation?

Automated reasoning

Idea: Use signed formulae

- F^v, where F is a formula and $v \in M$

  $\mathcal{A}, \beta \models F^v$ iff $\mathcal{A}(\beta)(F) = v$.

- $S{:}F$, where F is a formula and

  $\emptyset \neq S \subseteq M$ (set of truth values)

  $\mathcal{A}, \beta \models S{:}F$ iff $\mathcal{A}(\beta)(F) \in S$.

Semantic tableaux

For every $\emptyset \neq S \subseteq M$ and every logical operator f we have a tableau rule:

$$\frac{S{:}f(F_1, \dots, F_n)}{T(F_1, \dots, F_n)}$$

where $T(A_1, \dots, A_n)$ is a finite extended tableau containing only formulae of the form $S_i{:}F_i$.

Informally: Exhaustive list of conditions which ensure that the value of $f(F_1, \dots, F_n)$ is in S.

Example

Let Ł$_5$ be the 5-valued Łukasiewicz logic with $M = \{0, 1, 2, 3, 4\}$.

$\Rightarrow$	0	1	2	3	4
0	4	4	4	4	4
1	3	4	4	4	4
2	2	3	4	4	4
3	1	2	3	4	4
4	0	1	2	3	4

$$\{4\}(p \Rightarrow q)$$

$$\begin{array}{c|c|c|c|c} \hline \{0\}p & \{0,1\}p & \{0,1,2\}p & \{0,1,2,3\}p & \\ & \{1,2,3,4\}q & \{2,3,4\}q & \{3,4\}q & \{4\}q \end{array}$$

Labelling sets

Let $V \subseteq \mathcal{P}(M)$ be the set of all sets of truth values which are used for labelling the formulae.

Remarks:

1. In general not all subsets of truth values are used, so $V \neq \mathcal{P}(M)$.

2. Proof by contradiction:
   Goal: Prove that F is valid, i.e. $\mathcal{A}(\beta)(F) \in D$.
   We start from $(M\backslash D)$:F and build the tableau
   $\Rightarrow$ We assume that $(M\backslash D) \in V$.

3. Need to make sure that the new signs introduced by the tableau rules are in V.

Tableau rules: Soundness

$$\frac{S{:}f(F_1, \ldots, F_n)}{T(F_1, \ldots, F_n)}$$

where $T(F_1, \ldots, F_n)$ is a finite extended tableau containing only formulae of the form $S_i{:}F_i$.

$$S{:}f(F_1, \ldots, F_n)$$

$$\begin{array}{c|c|c|c}
\hline
S_{11}{:}C_{11} & S_{21}{:}C_{21} & \ldots & S_{q1}{:}C_{q1} \\
\ldots & \ldots & & \ldots \\
S_{1k_1}{:}C_{1k_1} & S_{2k_2}{:}C_{2k_2} & & S_{qk'}{:}C_{qk'}
\end{array}$$

where $C_{i,j} \in \{F_1, \ldots, F_n\}$

Tableau rules: Soundness

$$\frac{S{:}f(F_1, \ldots, F_n)}{T(F_1, \ldots, F_n)}$$

where $T(F_1, \ldots, F_n)$ is a finite extended tableau containing only formulae of the form $S_i{:}F_i$.

$$\begin{array}{c|c|c|c}
\multicolumn{4}{c}{S{:}f(F_1, \ldots, F_n)} \\
\hline
S_{11}{:}C_{11} & S_{21}{:}C_{21} & \ldots & S_{q1}{:}C_{q1} \\
\ldots & \ldots & & \ldots \\
S_{1k_1}{:}C_{1k_1} & S_{2k_2}{:}C_{2k_2} & & S_{qk'}{:}C_{qk'}
\end{array}$$

where $C_{i,j} \in \{F_1, \ldots, F_n\}$

For every $\mathcal{A}, \beta$: $\mathcal{A}(\beta)(F) \in S$ then there exists i such that for all j: $\mathcal{A}(\beta)(C_{ij}) \in S_{ij}$.

Tableau rules: Soundness

$$\begin{array}{c|c|c|c}
\multicolumn{4}{c}{S{:}f(F_1, \ldots, F_n)} \\
\hline
S_{11}{:}C_{11} & S_{21}{:}C_{21} & \ldots & S_{q1}{:}C_{q1} \\
\ldots & \ldots & & \ldots \\
S_{1k_1}{:}C_{1k_1} & S_{2k_2}{:}C_{2k_2} & & S_{qk'}{:}C_{qk'}
\end{array}$$

where $C_{i,j} \in \{F_1, \ldots, F_n\}$

Every model of $S{:}f(F_1, \ldots, F_n)$ is also a model of the formulae on one of the branches

If there is no expansion rule for a premise: premise is unsatisfiable ($\mathcal{A}(\beta)(F) \notin S$ for all $\mathcal{A}, \beta$).

If $f(F_1, \ldots, F_n)$ satisfiable then there is an expansion rule.

$\mathcal{L}_3$: Tableau rules for $\wedge$

$$\frac{\{1\}A \wedge B}{\begin{array}{c}\{1\}A \\ \{1\}B\end{array}} \qquad \frac{\{u\}A \wedge B}{\begin{array}{c|c|c}\{u\}A & \{u\}B & \{u\}A \\ \{1\}B & \{1\}A & \{u\}B\end{array}} \qquad \frac{\{0\}A \wedge B}{\{0\}A | \{0\}B} \qquad \frac{\{u, 0\}A \wedge B}{\{u, 0\}A | \{u, 0\}B}$$

$\mathcal{L}_3$: Tableau rules for $\vee$

$$\frac{\{1\}A \vee B}{\{1\}A \mid \{1\}B} \qquad \frac{\{u\}A \vee B}{\begin{array}{c|c} \{u,0\}A & \{u\}A \\ \{u\}B & \{u,0\}B \end{array}} \qquad \frac{\{0\}A \vee B}{\begin{array}{c} \{0\}A \\ \{0\}B \end{array}}$$

$$\frac{\{u,0\}A \vee B}{\begin{array}{c} \{u,0\}A \\ \{u,0\}B \end{array}}$$

$\mathcal{L}_3$: Tableau rules for $\neg, \sim$

$$\frac{\{1\} \sim A}{\{u,0\}A} \qquad \frac{\{0\} \sim A}{\{1\}A} \qquad \frac{\{u\} \sim A}{} \qquad \frac{\{u,0\} \sim A}{\{1\}A}$$

$$\frac{\{1\}\neg A}{\{0\}A} \qquad \frac{\{0\}\neg A}{\{1\}A} \qquad \frac{\{u\}\neg A}{\{u\}A} \qquad \frac{\{u,0\}\neg A}{\{1\}A|\{u\}A}$$

$\mathcal{L}_3$: Tableau rules for $\supset$

$$\frac{\{1\}A \supset B}{\{u,0\}A \mid \{1\}B} \qquad \frac{\{0\}A \supset B}{\begin{array}{c}\{1\}A\\ \{0\}B\end{array}} \qquad \frac{\{u\}A \supset B}{\begin{array}{c}\{1\}A\\ \{u\}B\end{array}} \qquad \frac{\{u,0\}A \supset B}{\begin{array}{c}\{1\}A\\ \{u,0\}B\end{array}}$$

$\mathcal{L}_3$: Tableau rules for $\exists$

$$\frac{\{1\}\exists x A(x)}{\{1\}A(f(y_1,\ldots,y_k))} \qquad \frac{\{0\}\exists x A(x)}{\{0\}A(z)} \qquad \frac{\{u\}\exists x A(x)}{\begin{array}{c}\{u\}A(f(y_1,\ldots y_k))\\ \{u,0\}A(z)\end{array}} \qquad \frac{\{u,0\}\exists x A(x)}{\{u,0\}A(z)}$$

where

- z is a new free variable
- $y_1,\ldots,y_k$ are the free variables in $\exists x A(x)$
- f is a new function symbol

$\mathcal{L}_3$: Tableau rules for $\forall$

$$\frac{\{1\}\forall x A(x)}{\{1\}A(z)} \quad \frac{\{0\}\forall x A(x)}{\{0\}A(f(y_1,\dots,y_k))} \quad \frac{\{u\}\forall x A(x)}{\begin{array}{c}\{u\}A(f(y_1,\dots y_k))\\ \{u,1\}A(z)\end{array}} \quad \frac{\{u,0\}\forall x A(x)}{\{u,0\}A(f(y_1,\dots,y_k))}$$

where

- z is a new free variable
- $y_1,\dots,y_k$ are the free variables in $\forall x A(x)$
- f is a new function symbol

Tableaux

A tableau for a finite set For of signed formulae is constructed as follows:

- A linear tree, in which each formula in For occurs once is a tableau.
- Let T be a tableau for For und P a path in T, which contains a signed formula $S{:}F$.

  Assume that there exists a tableau rule with premise $S{:}F$. If $E_1, \ldots, E_n$ are the possible conclusions of the tableau rule (under the corresponding restrictions in case of quantified formulae) then T is exteded with n linear subtrees containing the signed formulae from E_i (respectively), in arbitrary order.

  The tree obtained this way is again a tableau for For.

Closed Tableaux

A path P in a tableau T is closed if:

- P contains complementary formulae, i.e. there exists a substitution σ and there exists signed formulae $S_1{:}F_1, \ldots, S_k{:}F_k$ occurring of the branch such that:
  - $F_1\sigma = \cdots = F_n\sigma$
  - $S_1 \cap \cdots \cap S_n = \emptyset$, or
- P contains a signed formula $S{:}F$ for which no expansion rule can be applied and F is not atomic.

A path which is not closed is called open.

Closed Tableaux

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  - $F_1\sigma = \cdots = F_n\sigma$
  - $S_1 \cap \cdots \cap S_n = \emptyset$, or
- P contains a signed formula $S{:}F$ for which no expansion rule can be applied and F is not atomic.

A path which is not closed is called open.

A tableau is closed if every path can be closed with the same substitution.

Otherwise the tableau is called open.

Soundness and completeness

Given an signature Σ, by Σ^{sko} we denote the result of adding infinitely many new Skolem function symbols which we may use in the rules for quantifiers.

Let $\mathcal{A}$ be a Σ^{sko}-interpretation, T a tableau, and β a variable assignment over $\mathcal{A}$.

T is called $(\mathcal{A}, \beta)$-valid, if there is a path P_β in T such that $\mathcal{A}, \beta \models F$, for each formula F on P_β.

T is called satisfiable if there exists a structure $\mathcal{A}$ such that for each assignment β the tableau T is $(\mathcal{A}, \beta)$-valid.
(This implies that we may choose P_β depending on β.)

Soundness and completeness

Theorem (Soundness of the tableau calculus for $\mathcal{L}_3$)
Let F be a $\mathcal{L}_3$-formula without free variables. If there exists a closed tableau T for $\{U, F\}F$, then F is an $\mathcal{L}_3$-tautology (it is valid).

Theorem (Refutational completeness)
Let F be a $\mathcal{L}_3$-tautology. Then we can construct a closed tableau for $\{U, F\}F$. (The order in which we apply the expansion rules is not important).