Non-classical logics

Lecture 11: Modal logics (Part 1)

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History and Motivation

Extensions of classical logic by means of new logical operators

Modal logic

- modal operators $\Box$, $\Diamond$

meaning of $\Box A$	meaning of $\Diamond A$
A is necessarily true	A is possibly true
An agent believes A	An agent thinks A is possible
A is always true	A is sometimes true
A should be the case	A is allowed
A is provable	A is not contradictory

History and Motivation

Logics related to modal logic

Dynamic logic of programs

Operators:

$\boxed{\alpha}A$: A holds after every run of the (non-deterministic) process α

$\langle\alpha\rangle A$: A holds after some run of the (non-deterministic) process α

History and Motivation

Logics related to modal logic

Temporal logic

$\Box A$:	A holds always (in the future)
$\Diamond A$:	A holds at some point (in the future)
$\circ A$:	A holds at the next time point (in the future)
A until B	A must remain true at all following time points until B becomes true

History and Motivation

Extensions of classical logic: **Modal logic and related logics**

Very rich history.

Antiquity and middle ages

John Duns Scotus (1266 - 1308)

Reasoned informally in a modal manner, mainly to analyze statements about possibility and necessity.

William of Ockham (1288 - 1348)

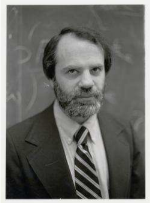
In addition to his work on De Morgan's Laws and ternary logic, he also analyzed statements about possibility and necessity.

Beginning of modern modal logic

Clarence Irving Lewis (1883-1964)

founded modern modal logic in his 1910 Harvard thesis.

Saul Kripke (1940-)



In 1959, **Saul Kripke** (then a 19-year old Harvard student) introduced the possible-worlds semantics for modal logics.

Ruth C. Barcan, later Ruth Barcan Marcus (1921-2012)

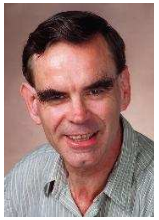
Developed the first axiomatic systems of quantified modal logic.

Temporal logic and dynamic logic

Arthur Norman Prior (1914-1969)

Created modern temporal logic in 1957

Vaughan Pratt (1944-)



Introduced dynamic logic in 1976.

Amir Pnueli (1941-2009)

In 1977, proposed using temporal logic to formalise the behaviour of continually operating concurrent programs.

Modal logic

In classical logic, it is only important whether a formula is true In modal logic, it is also important in which

- way
- mode
- state

a formula is true

Modal logic

A formula (a proposition) is

- necessarily / possibly true
- true today / tomorrow
- believed / known
- true before / after an action / the execution of a program

New operator $\Box$ / $\Diamond$

(or families of such operators)

Propositional modal logic

- Syntax
- Inference systems and proofs

- Semantics

Soundness and completeness

Decidability

Literature

Modal, temporal and dynamic logic

- Bull and Segerberg “Basic modal logic”. In Handbook of Philosophical Logic,
- Goldblatt, R. “Logics of time and computation”. CSLI Series, 1987
- Hughes, G.E. and Cresswell, M.J.
  - A new introduction to modal logic, 1st ed., Routledge, 1996.
  - A companion to modal logic, Methuen, 1985.
  - Introduction to modal logic (repr. 1990), Routledge, 1972.
- Huth, M. and Ryan, M. Logic in Computer Science: Modelling and reasoning about systems, Cambridge University Press, 2000
- Fitting, M. “Basic modal logic”. In Handbook of Logic in Artificial Intelligence and Logic Programming, Vol 1: Logical Foundations. 368-448
- Fitting, M. “Proof methods for modal and intuitionistic logics”. Kluwer, 1983.
- Fitting, M. and Mendelsohn, R. “First-order modal logic”. Kluwer, 1998

Literature

Modal and temporal logic

- Stirling, C. "Modal and temporal logics". In Handbook of Logics in Computer Science, Vol 2: Background: Computational Structures (Gabbay, D. and Abramski, S. and Maibaum, T.S.E. eds), pages 478-563, Clarendon Press, 1992.
- Stirling, C. "Modal and temporal properties of processes". Springer Texts in computer science, 2001.
- Emerson, E.A. "Temporal and modal logic". Handbook of Theoretical Computer Science, 1990.
- Kroeger, F. "Temporal logic of programs". EATCS monographs on theoretical computer science, Springer, 1987.
- Clarke, E.N., Emerson, E.A., Sistla, A.P. "Automatic verification of finite-state concurrent systems using temporal logic specifications". ACM Transactions on Programming Languages and Systems (TOPLAS) 8(2): 244-263

Literature

Modal and temporal logic

- Harel, D., Kozen, D. and Tiuryn, J. “Dynamic logic”. MIT Press, 2000

Syntax

- propositional variables
- logical symbols: $\{\vee, \wedge, \neg, \rightarrow, \leftrightarrow, \Box, \Diamond\}$

Propositional Variables

Let Π be a set of propositional variables.

We use letters P, Q, R, S, to denote propositional variables.

Propositional Formulas

F_Π is the set of propositional formulas over Π defined as follows:

$$
\begin{array}{lcll}
F, G, H & ::= & \bot & \text{(falsum)} \\
& \mid & \top & \text{(verum)} \\
& \mid & P, \quad P \in \Pi & \text{(atomic formula)} \\
& \mid & \neg F & \text{(negation)} \\
& \mid & (F \land G) & \text{(conjunction)} \\
& \mid & (F \lor G) & \text{(disjunction)} \\
& \mid & (F \rightarrow G) & \text{(implication)} \\
& \mid & (F \leftrightarrow G) & \text{(equivalence)} \\
& \mid & \Box F & \\
& \mid & \Diamond F &
\end{array}
$$

Informal Interpretations of $\Box$

$\Box F$ can mean:

- F is necessarily true
- F is always true (in future states/words)
- an agent a believes F
- an agent a knows F
- F is true after all possible executions of a program p

Informal Interpretations of $\Box$

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Notation: If necessary write

$$\Box_a F, \Box_p F, [a]F, [p]F$$

instead of $\Box F$.

Informal Interpretations of $\Box$, $\Diamond$

meaning of $\Box A$	meaning of $\Diamond A = \neg\Box\neg A$
A is necessarily true	A is possibly true
A is always true	A is sometimes true
Agent a believes A	Agent A thinks A is possible
Agent a believes A	A is consistent with a's beliefs
Agent a knows A	a does not know $\neg A$
A should be the case	A is allowed
A is provable	A is not contradictory
A holds after every run of the (non-deterministic) program p	A is true after at least one possible execution of program p

The Wise-Men Puzzle

There are three wise men, three red hats, and two white hats. The king puts a hat on each of the wise men in such a way that they are not able to see their own hat.

He then asks each one in turn whether he knows the color of his hat.

The first man says he does not know.

The second man says he does not know either.

What does the third man say?

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The second man says he does not know either.

What does the third man say?

- if there is only one red hat, he will answer "red"
- if there are two red hats, the wearers will know this after the question is repeated
- if there are three red hats, the question has to be is repeated once more

The Muddy-Children Puzzle

Three children are playing in the garden and some of the children get mud on their foreheads.

Each child can see the mud on others only.

Now consider two scenarios:

- The father repeatedly asks "Does any of you know whether you have mud on your forehead?".

  All children answer "no" the first time, and continue to answer "no" to repetitions of the same question.

- The father tells the children that at least one of them is muddy and repeatedly asks "Does any of you know whether you have mud on your forehead?". After the question has been asked ≤ 3 times, the muddy children will answer "yes."

The Muddy-Children Puzzle

Consider the second scenario.

k = 1. There is only one muddy child, which will answer "yes" because of the father's statement.

k = 2. If two children, call them *a* and *b*, are muddy, they both answer "no" the first time. But both *a* and *b* then reason that the other muddy child must have seen someone with mud on his forehead, and hence answer "yes" the second time.

k = 3. Let *a*, *b*, and *c* be the muddy children. Everybody answers "no" the first two times. But then *a* reasons that if *b* and *c* are the only muddy children they would have answered "yes" the second time (based on the argument for the case $k = 2$). Since they answered "no," *a* further reasons, they must have seen a third child with mud, which must be me. Children *b* and *c* reason in the same way, and all three children answer "yes" the third time.

The Muddy-Children Puzzle

Note that the father's announcement makes it common knowledge among the children that at least one child is muddy.

Generalization

A group of children is playing in the garden and some of the children, say k of them, get mud on their foreheads. Each child can see the mud on others only. Note that if $k > 1$, then every child can see another with mud on its forehead.

The father tells the children that at least one of them is muddy and repeatedly asks "Does any of you know whether you have mud on your forehead?".

After the question has been asked k times, the k muddy children will answer "yes".

Formalizing the Wise-Men Puzzle

Notation:

r_i means "man i wears a red hat"

w_i means "man i wears a white hat"

The situation can be described by the following formulae:

$$\{(r_1 \vee r_2 \vee r_3), \neg(r_1 \wedge w_1), \neg(r_2 \wedge w_2), \neg(r_3 \wedge w_3), \neg w_1 \leftrightarrow r_1, \neg w_2 \leftrightarrow r_2, \neg w_3 \leftrightarrow r_3$$

$$(r_1 \rightarrow \Box_2 r_1), (w_1 \rightarrow \Box_2 w_1), (r_1 \rightarrow \Box_3 r_1), (w_1 \rightarrow \Box_3 w_1),$$

$$(r_2 \rightarrow \Box_1 r_2), (w_2 \rightarrow \Box_1 w_2), (r_2 \rightarrow \Box_3 r_2), (w_2 \rightarrow \Box_3 w_2),$$

$$(r_3 \rightarrow \Box_1 r_3), (w_3 \rightarrow \Box_1 w_3), (r_3 \rightarrow \Box_2 r_3), (w_3 \rightarrow \Box_2 w_3)\}$$

Facts:

$$\neg\Box_1 r_1, \neg\Box_2 r_2$$

Formalization

- Formalize the properties of $\Box_i$
- Entail the truth of certain formulae

Proof Calculi/Inference systems and proofs

Inference systems Γ (proof calculi) are sets of tuples

$$(F_1, \ldots, F_n, F_{n+1}), \;\; n \geq 0,$$

called inferences or inference rules, and written

$$\frac{\overbrace{F_1 \;\ldots\; F_n}^{\text{premises}}}{\underbrace{F_{n+1}}_{\text{conclusion}}}\;.$$

Inferences with 0 premises are also called axioms.

Proofs

A proof in Γ of a formula F from a a set of formulas N (called assumptions) is a sequence $F_1, \ldots, F_k$ of formulas where

(i) $F_k = F$,

(ii) for all $1 \leq i \leq k$: $F_i \in N$, or else there exists an inference $(F_{i_1}, \ldots, F_{i_{n_i}}, F_i)$ in Γ, such that $0 \leq i_j < i$, for $1 \leq j \leq n_i$.

Provability

Provability $\vdash_\Gamma$ of F from N in Γ:

$N \vdash_\Gamma F \;\; :\Leftrightarrow\;$ there exists a proof Γ of F from N.

Inference system for modal logic

Acceptable axioms:

- All axioms of propositional logic (e.g. $p \vee \neg p$)
- $(\Box A \wedge \Box(A \to B)) \to \Box B$
- $\Box(A \to B) \to (\Box A \to \Box B)$

Inference system for modal logics

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Acceptable inference rules

$$\frac{A \quad A \to B}{B}$$ [Modus ponens]

$$\frac{A}{\Box A}$$ [Necessitation]

Remark: Accepting the last inference rule is not the same with accepting $A \to \Box A$ as an axiom!

Example of proof

Task: Check whether the following can be proved the inference system of modal logic indicated on page 33:

$\{\Box(A \land B)\} \vdash \Box A \land \Box B$

Example of proof

Task: Check whether $\{\Box(A \wedge B)\} \vdash \Box A \wedge \Box B$

1.	$\Box(A \wedge B)$	premise
2.	$A \wedge B \rightarrow A$	theorem prop. logic
3.	$A \wedge B \rightarrow B$	theorem prop. logic
4.	$\Box(A \wedge B \rightarrow A)$	necessitation 2.
5.	$\Box(A \wedge B \rightarrow B)$	necessitation 3.
6.	$\Box(A \wedge B) \wedge \Box(A \wedge B \rightarrow A)$	theorem prop. logic (1, 5)
7.	$\Box(A \wedge B) \wedge \Box(A \wedge B \rightarrow B)$	theorem prop. logic (1, 6)
8.	$\Box(A \wedge B) \wedge \Box(A \wedge B \rightarrow A) \rightarrow \Box A$	Ax1
9.	$\Box A$	MP (6, 8)
10	$\Box(A \wedge B) \wedge \Box(A \wedge B \rightarrow B) \rightarrow \Box B$	Ax1
11	$\Box B$	MP (7, 9)
12	$\Box A \wedge \Box B$	theorem prop. logic

The modal system K

Axioms:

- All axioms of propositional logic (e.g. $p \vee \neg p$)
- $\Box(A \to B) \to (\Box A \to \Box B)$ (K)

Inference rules

$$\frac{A \qquad A \to B}{B}$$ [Modus ponens]

$$\frac{A}{\Box A}$$ [G]

Some systems of modal logic

System	*Description*
T	$K + \Box A \to A$
D	$K + \Box A \to \Diamond A$
B	$T + \neg A \to \Box \neg \Box A$
$S4$	$T + \Box A \to \Box \Box A$
$S5$	$T + \neg \Box A \to \Box \neg \Box A$
$S4.2$	$S4 + \diamond \Box A \to \Box \Diamond A$
$S4.3$	$S4 + \Box(\Box(A \to B)) \vee \Box(\Box(B \to A))$
C	$K + \frac{A \to B}{\Box(A \to B)}$ instead of (G).

Semantics of modal logic

Two classes of models have been studied so far.

- Modal algebras
- Kripke models

Semantics of modal logic

Modal algebras $(B, \vee, \wedge, \neg, \rightarrow, \leftrightarrow, 0, 1, \Box, \Diamond)$ where

- $(B, \vee, \wedge, \neg, 0, 1)$ Boolean algebra, i.e. satisfies the following conditions:

$x \wedge y = y \wedge x$	$x \vee y = y \vee x$
$x \wedge (y \wedge z) = (x \wedge y) \wedge z$	$x \vee (y \vee z) = (x \vee y) \vee z$
$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
$x \wedge x = x$	$x \vee x = x$
$x \wedge (x \vee y) = x$	$x \vee (x \wedge y) = x$
$x \wedge 1 = x$	$x \vee 0 = x$
$x \wedge 0 = 0$	$x \vee 1 = 1$
$x \vee \neg x = 1$	$x \wedge \neg x = 0$

- $\rightarrow, \leftarrow$ derived operations: $x \rightarrow y := \neg x \vee y; \quad x \leftrightarrow y := (x \rightarrow y) \wedge (y \rightarrow x)$

  $\Diamond x = \neg \Box \neg x$

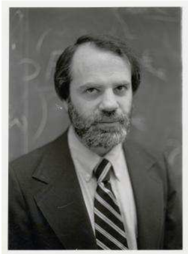
- $\Box$ has additional properties e.g. $\Box(x \wedge y) = \Box x \wedge \Box y$

Kripke Frames and Kripke Structures

Introduced by Saul Aaron Kripke in 1959.

Much less complicated and better suited to automated reasoning than modal algebras.

Saul Aaron Kripke



Born	1940 in Omaha (US)
First publication:	A Completeness Theorem in Modal Logic The Journal of Symbolic Logic, 1959
Studied at:	Harvard, Princeton, Oxford and Rockefeller University
Positions:	Harvard, Rockefeller, Columbia, Cornell, Berkeley, UCLA, Oxford since 1977 Professor at Princeton University since 1998 Emeritus at Princeton University

Kripke Frames and Kripke Structures

Definition. A Kripke frame $F = (S, R)$ consists of

- a non-empty set S (of possible worlds / states)
- an accessibility relation $R \subseteq S \times S$

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Definition. A Kripke frame $F = (S, R)$ consists of

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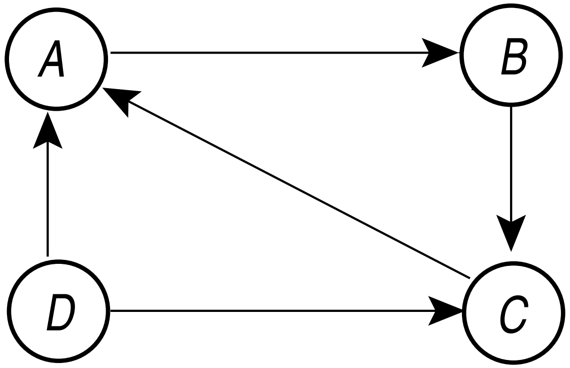
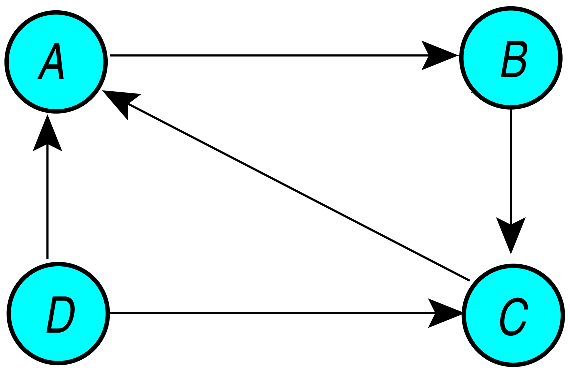
Definition. A Kripke structure $K = (S, R, \mathcal{I})$ consists of

- a Kripke frame $F = (S, R)$
- an interpretation $\mathcal{I} : \Pi \times S \to \{1, 0\}$

Example of Kripke frame

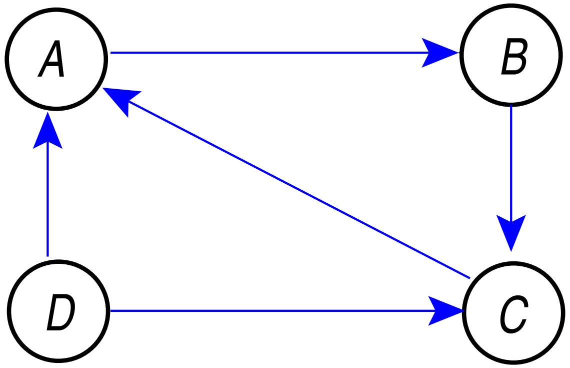


Example of Kripke frame



Set of possible worlds (states): $S = \{A, B, C, D\}$

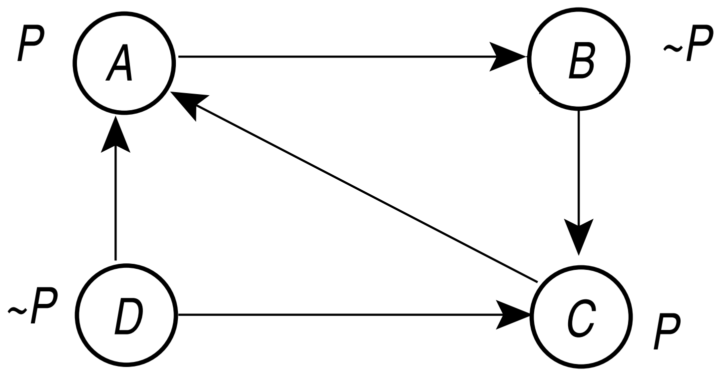
Example of Kripke frame



Set of possible worlds (states): $S = \{A, B, C, D\}$

Accessibility relation: $R = \{(A, B), (B, C), (C, A), (D, A), (D, C)\}$

Example of Kripke structure



Set of possible worlds (states): $S = \{A, B, C, D\}$

Accessibility relation: $R = \{(A, B), (B, C), (C, A), (D, A), (D, C)\}$

Interpretation: $\mathcal{I} : \Pi \times S \rightarrow \{0, 1\}$

$\mathcal{I}(P, A) = 1, \mathcal{I}(P, B) = 0, \mathcal{I}(P, C) = 1, \mathcal{I}(P, D) = 0$

Notation Instead of $(A, B) \in R$ we will sometimes write ARB.

Notation

$K = (S, R, I)$

Instead of writing $(s, t) \in R$ we will sometimes write sRt.

Modal logic: Semantics

Given: Kripke structure $K = (S, R, I)$

Valuation:

$val_K(p)(s) = I(p, s)$ for $p \in \Pi$

val_K defined for propositional operators in the same way as in classical logic

$$val_K(\Box A)(s) = \begin{cases} 1 & \text{if } val_K(A)(s') = 1 \text{ for all } s' \in S \text{ with } sRs' \\ 0 & \text{otherwise} \end{cases}$$

$$val_K(\Diamond A)(s) = \begin{cases} 1 & \text{if } val_K(A)(s') = 1 \text{ for at least one } s' \in S \text{ with } sRs' \\ 0 & \text{otherwise} \end{cases}$$

Models, Validity, and Satisfiability

$\mathcal{F} = (S, R)$, $\mathcal{K} = (S, R, I)$

F is true in $\mathcal{K}$ at a world $s \in S$:

$$(\mathcal{K}, s) \models F :\Leftrightarrow \text{val}_{\mathcal{K}}(F)(s) = 1$$

F is true in $\mathcal{K}$

$$\mathcal{K} \models F :\Leftrightarrow (\mathcal{K}, s) \models F \text{ for all } s \in S$$

F is true in the frame $\mathcal{F} = (S, R)$

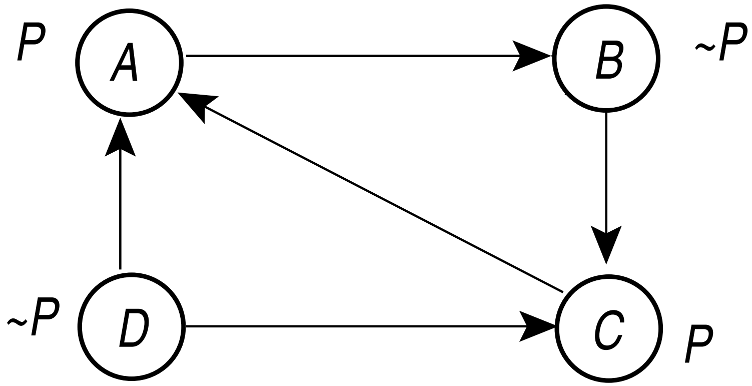
$$\mathcal{F} \models F :\Leftrightarrow (\mathcal{K}_{\mathcal{F}}) \models F \text{ for all Kripke structures } \mathcal{K}_{\mathcal{F}} = (S, R, I')$$

defined on frame $\mathcal{F}$

If Φ is a class of frames, F is true (valid) in Φ

$$\Phi \models F :\Leftrightarrow \mathcal{F} \models F \text{ for all } \mathcal{F} \in \Phi.$$

Example for evaluation



$(\mathcal{K}, A) \models P$ $\quad (\mathcal{K}, B) \models \neg P$ $\quad (\mathcal{K}, C) \models P$ $\quad (\mathcal{K}, D) \models \neg P$

$(\mathcal{K}, A) \models \Box\neg P$ $\quad (\mathcal{K}, B) \models \Box P$ $\quad (\mathcal{K}, C) \models \Box P$ $\quad (\mathcal{K}, D) \models \Box P$

$(\mathcal{K}, A) \models \Box\Box P$ $\quad (\mathcal{K}, B) \models \Box\Box P$ $\quad (\mathcal{K}, C) \models \Box\Box\neg P$ $\quad \ldots$

Entailment and Equivalence

In classical logic we proved:

Proposition:
F entails G iff $(F \to G)$ is valid

Does such a result hold in modal logic?

Entailment

In classical logic we proved:

Proposition:
$F \models G$ iff $(F \rightarrow G)$ is valid

Does such a result hold in modal logic?

Need to define what $F \models G$ means