Non-classical logics

Lecture 12 + 13: Modal logics (Part 2)

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Until now

- History and Motivation
- Propositional modal logic

    Syntax

    Inference systems and proofs

    Semantics

Semantics of modal logic

Two classes of models have been studied so far.

- Modal algebras
- Kripke models

Kripke Frames and Kripke Structures

Introduced by Saul Aaron Kripke in 1959.

Much less complicated and better suited to automated reasoning than modal algebras.

Kripke Frames and Kripke Structures

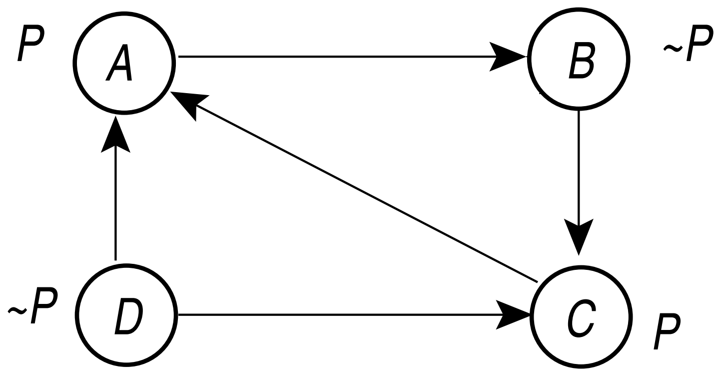
Definition. A Kripke frame $F = (S, R)$ consists of

- a non-empty set S (of possible worlds / states)
- an accessibility relation $R \subseteq S \times S$

Definition. A Kripke structure $K = (S, R, \mathcal{I})$ consists of

- a Kripke frame $F = (S, R)$
- an interpretation $\mathcal{I} : \Pi \times S \to \{1, 0\}$

Example of Kripke structure



Set of possible worlds (states): $S = \{A, B, C, D\}$

Accessibility relation: $R = \{(A, B), (B, C), (C, A), (D, A), (D, C)\}$

Interpretation: $\mathcal{I} : \Pi \times S \rightarrow \{0, 1\}$

$\mathcal{I}(P, A) = 1, \mathcal{I}(P, B) = 0, \mathcal{I}(P, C) = 1, \mathcal{I}(P, D) = 0$

Notation Instead of $(A, B) \in R$ we will sometimes write ARB.

Modal logic: Semantics

Given: Kripke structure $K = (S, R, I)$

Valuation:

$val_K(p)(s) = I(p, s)$ for $p \in \Pi$

val_K defined for propositional operators in the same way as in classical logic

$$val_K(\Box A)(s) = \begin{cases} 1 & \text{if } val_K(A)(s') = 1 \text{ for all } s' \in S \text{ with } sRs' \\ 0 & \text{otherwise} \end{cases}$$

$$val_K(\Diamond A)(s) = \begin{cases} 1 & \text{if } val_K(A)(s') = 1 \text{ for at least one } s' \in S \text{ with } sRs' \\ 0 & \text{otherwise} \end{cases}$$

Models, Validity, and Satisfiability

$\mathcal{F} = (S, R)$, $\mathcal{K} = (S, R, I)$

F is true in $\mathcal{K}$ at a world $s \in S$:

$$(\mathcal{K}, s) \models F :\Leftrightarrow \mathrm{val}_{\mathcal{K}}(F)(s) = 1$$

F is true in $\mathcal{K}$

$$\mathcal{K} \models F :\Leftrightarrow (\mathcal{K}, s) \models F \text{ for all } s \in S$$

F is true in the frame $\mathcal{F} = (S, R)$

$$\mathcal{F} \models F :\Leftrightarrow (\mathcal{K}_{\mathcal{F}}) \models F \text{ for all Kripke structures } \mathcal{K}_{\mathcal{F}} = (S, R, I')$$

$$\text{defined on frame } \mathcal{F}$$

If Φ is a class of frames, F is true (valid) in Φ

$$\Phi \models F :\Leftrightarrow \mathcal{F} \models F \text{ for all } \mathcal{F} \in \Phi.$$

Entailment and Equivalence

In classical logic we proved:

Proposition:
F entails G iff $(F \to G)$ is valid

Does such a result hold in modal logic?

Entailment

In classical logic we proved:

Proposition:

$F \models G$ iff $(F \to G)$ is valid

Does such a result hold in modal logic?

Need to define what $F \models G$ means

Entailment

Goal: definition for $N \models F$, where N is a family of modal formulae

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Tentative 1:

$N \models_G F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$:

$$\text{If } \mathcal{K} \models G \text{ for every } G \in N \text{ then } \mathcal{K} \models F$$

Entailment

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"global entailment"

Example

$N \models_G F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$:

$$\text{If } \mathcal{K} \models G \text{ for every } G \in N \text{ then } \mathcal{K} \models F$$

Task: Show that $P \models_G \Box P$

Proof: Let $\mathcal{K} = (S, R, I)$ be a Kripke structure.

Assume that $\mathcal{K} \models P$, i.e. for every $s \in S$ we have $(\mathcal{K}, s) \models P$.

Then it follows that for every $s \in S$ we have $(\mathcal{K}, s) \models \Box P$.

By the definition of $\models_G$ it follows that $P \models_G \Box P$.

Example

$N \models_G F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$:

$$\text{If } \mathcal{K} \models G \text{ for every } G \in N \text{ then } \mathcal{K} \models F$$

Proved: $P \models_G \Box P$

Question: Is it true that $P \rightarrow \Box P$ is true in all Kripke structures?

Answer: Let $\mathcal{K} = (S, R, I)$, where

$S = \{s_1, s_2\}$, $R = \{(s_1, s_2)\}$, $I(P, s_1) = 1, I(P, s_2) = 0$.

Then $(\mathcal{K}, s_1) \models P$, $(\mathcal{K}, s_1) \not\models \Box p$.

Hence $(\mathcal{K}, s_1) \not\models P \rightarrow \Box P$.

Entailment

Goal: definition for $N \models F$, where N is a family of modal formulae

Tentative 2:

$N \models_L F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$ and every $s \in S$:

$$\text{If } (\mathcal{K}, s) \models G \text{ for every } G \in N \text{ then } (\mathcal{K}, s) \models F$$

Entailment

Goal: definition for $N \models F$, where N is a family of modal formulae

Tentative 2:

$N \models_L F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$ and every $s \in S$:

$$\text{If } (\mathcal{K}, s) \models G \text{ for every } G \in N \text{ then } (\mathcal{K}, s) \models F$$

"local entailment"

Entailment

$N \models_G F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$:

$$\text{If } \mathcal{K} \models G \text{ for every } G \in N \text{ then } \mathcal{K} \models F$$

$N \models_L F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$ and every $s \in S$:

$$\text{If } (\mathcal{K}, s) \models G \text{ for every } G \in N \text{ then } (\mathcal{K}, s) \models F$$

Remark: The two entailment relations are different

$P \models_G \Box P$ (was shown before)

$P \not\models_L \Box P$

Entailment

$N \models_G F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$:

$$\text{If } \mathcal{K} \models G \text{ for every } G \in N \text{ then } \mathcal{K} \models F$$

$N \models_L F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$ and every $s \in S$:

$$\text{If } (\mathcal{K}, s) \models G \text{ for every } G \in N \text{ then } (\mathcal{K}, s) \models F$$

Remark: The two entailment relations are different

$P \models_G \Box P$ (was shown before)

$P \not\models_L \Box P$

Proof: Let $\mathcal{K} = (S, R, I)$, where

$S = \{s_1, s_2\}$, $R = \{(s_1, s_2)\}$, $I(P, s_1) = 1, I(P, s_2) = 0$.

Then $(\mathcal{K}, s_1) \models P$, but $(\mathcal{K}, s_1) \not\models \Box P$. Hence, $P \not\models_L \Box P$.

Entailment

Theorem (The deduction theorem) The following are equivalent:

(1) $F \models_L G$

(2) $\{F, \neg G\}$ is unsatisfiable

(3) $\models (F \rightarrow G)$

(4) $\models_L (F \rightarrow G)$

Proof. $F \models_L G$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$ and every $s \in S$:

If $(\mathcal{K}, s) \models F$ then $(\mathcal{K}, s) \models G$

iff there is no Kripke structure $\mathcal{K} = (S, R, I)$ and no $s \in S$ with $(\mathcal{K}, s) \models F \wedge \neg G$

iff $\{F, \neg G\}$ is unsatisfiable

From propositional logic we know that $\{F, \neg G\}$ is unsatisfiable iff $F \rightarrow G$ is valid. This happens iff $\models_L F \rightarrow G$

Modal Logic: Valid Formulae

Valid:

- $\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$
- $(\Box P \wedge \Box(P \rightarrow Q)) \rightarrow \Box Q$
- $(\Box P \vee \Box Q) \rightarrow \Box(P \vee Q)$
- $(\Box P \wedge \Box Q) \leftrightarrow \Box(P \wedge Q)$
- $\Box P \leftrightarrow \neg\Diamond\neg P$
- $\Diamond(P \vee Q) \leftrightarrow (\Diamond P \vee \Diamond Q)$
- $\Diamond(P \wedge Q) \rightarrow (\Diamond P \wedge \Diamond Q)$

Modal Logic: Valid Formulae

Valid:

- $\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$
- $(\Box P \wedge \Box(P \rightarrow Q)) \rightarrow \Box Q$
- $(\Box P \vee \Box Q) \rightarrow \Box(P \vee Q)$
- $(\Box P \wedge \Box Q) \leftrightarrow \Box(P \wedge Q)$
- $\Box P \leftrightarrow \neg\Diamond\neg P$
- $\Diamond(P \vee Q) \leftrightarrow (\Diamond P \vee \Diamond Q)$
- $\Diamond(P \wedge Q) \rightarrow (\Diamond P \wedge \Diamond Q)$

Not valid:

- $\Box(P \vee Q) \rightarrow (\Box P \vee \Box Q)$
- $(\Diamond P \wedge \Diamond Q) \rightarrow \Diamond(P \wedge Q)$

Modal Logic: Valid Formulae

Not valid: $\Box(P \vee Q) \rightarrow (\Box P \vee \Box Q)$

[explanations on the blackboard]

Exercises

1. Show that $\Diamond T$ and the schema $\Box A \to \Diamond A$ have exactly the same models.

2. Exhibit a frame in which $\Box \perp$ is valid.

3. In any model $\mathcal{K}$,

   (i) if A is a tautology then $\mathcal{K} \models A$;

   (ii) if $\mathcal{K} \models A$ and $\mathcal{K} \models A \to B$, then $\mathcal{K} \models B$;

   (iii) if $\mathcal{K} \models A$ then $\mathcal{K} \models \Box A$.

Correspondence Theory

Correspondence Theory

Main questions:

Assume that we consider a set of frames for which the accessibility relation has certain properties. Is it the case that in all frames in this class a certain modal formula holds?

Given a modal formula. Can we describe the frames in which the formula holds, e.g. by specifying certain properties of the accessibility relation?

Example

Let ReflFrames be the class of all frames $\mathcal{F} = (S, R)$ in which R is reflexive.

Theorem. For every formula A, the formula $\Box A \rightarrow A$ is true in all frames $\mathcal{F} = (S, R) \in$ ReflFrames (i.e. in all frames $\mathcal{F} = (S, R)$ with R reflexive).

[Proof on the blackboard]

Example

Let ReflFrames be the class of all frames $\mathcal{F} = (S, R)$ in which R is reflexive.

Theorem. For every formula A, the formula $\Box A \to A$ is true in all frames in ReflFrames.

Theorem. If the formula $\Box A \to A$ is true in a frame $\mathcal{F} = (S, R)$ for every formula A, then R must be reflexive.

[Proof on the blackboard]

Conditions on R

The following is a list of properties of a binary relation R that are denned by first-order sentences.

1. Reflexive:	$\forall s \, (sRs)$
2. Symmetric:	$\forall s \forall t \, (sRt \rightarrow tRs)$
3. Serial:	$\forall s \exists t \, (sRt)$
4. Transitive:	$\forall s \forall t \forall u \, (sRt \wedge tRu \rightarrow sRu)$
5. Euclidean:	$\forall s \forall t \forall u \, (sRt \wedge sRu \rightarrow tRu)$
6. Partially functional:	$\forall s \forall t \forall u \, (sRt \wedge sRu \rightarrow t = u)$
7. Functional:	part. functional $+ \forall s \exists t (sRt)$
8. Weakly dense:	$\forall s \forall t (sRt \rightarrow \exists u \, (sRu \wedge uRt))$
9. Weakly connected:	$\forall s \forall t \forall u \, (sRt \wedge sRu \rightarrow tRu \vee t = u \vee uRt)$
10. Weakly directed:	$\forall s \forall t \forall u \, (sRt \wedge sRu \rightarrow \exists v (tRv \wedge uRv))$

List of schemata of modal formulae

Corresponding to the list of properties of R is a list of schemata:

1. $\Box A \to A$
2. $A \to \Box\Diamond A$
3. $\Box A \to \Diamond A$
4. $\Box A \to \Box\Box A$
5. $\Diamond A \to \Box\Diamond A$
6. $\Diamond A \to \Box A$
7. $\Diamond A \leftrightarrow \Box A$
8. $\Box\Box A \to \Box A$
9. $\Box(A \land \Box A \to B) \lor \Box(B \land \Box B \to A)$
10. $\Diamond\Box A \to \Box\Diamond A$

Correspondence theorems

Properties of R		**Axioms**
1. Reflexive:	$\forall s\ (sRs)$	$\Box A \to A$
2. Symmetric:	$\forall s \forall t\ (sRt \to tRs)$	$A \to \Box \Diamond A$
3. Serial:	$\forall s \exists t\ (sRt)$	$\Box A \to \Diamond A$
4. Transitive:	$\forall s \forall t \forall u\ (sRt \wedge tRu \to sRu)$	$\Box A \to \Box \Box A$
5. Euclidean:	$\forall s \forall t \forall u\ (sRt \wedge sRu \to tRu)$	$\Diamond A \to \Box \Diamond A$
6. Partially functional:	$\forall s \forall t \forall u\ (sRt \wedge sRu \to t = u)$	$\Diamond A \to \Box A$
7. Functional:	part. functional + $\forall s \exists t (sRt)$	$\Diamond A \leftrightarrow \Box A$
8. Weakly dense:	$\forall s \forall t (sRt \to \exists u\ (sRu \wedge uRt))$	$\Box \Box A \to \Box A$
9. Weakly connected:	$\forall s \forall t \forall u\ (sRt \wedge sRu \to tRu \vee t = u \vee uRt)$	$\Box(A \wedge \Box A \to B) \vee \Box(B \wedge \Box B \to A)$
10. Weakly directed:	$\forall s \forall t \forall u\ (sRt \wedge sRu \to \exists v (tRv \wedge uRv))$	$\Diamond \Box A \to \Box \Diamond A$

Theorem. Let $\mathcal{F} = (S, R)$ be a frame.

Then for each of the properties 1-10, if R satisfies the property, then the corresponding schema is valid in $\mathcal{F}$.

Theorem. If a frame $\mathcal{F} = (S, R)$ validates any one of the schemata 1-10, then R satisfies the corresponding property.

Correspondence theorems

Theorem. Let $\mathcal{F} = (S, R)$ be a frame.

Then for each of the properties 1-10, if R satisfies the property, then the corresponding schema is valid in $\mathcal{F}$.

Proof. We illustrate with the case of transitivity. Suppose that R is transitive. Let $\mathcal{K}$ be any model on $\mathcal{F}$.

To show that $\mathcal{K} \models \Box A \to \Box\Box A$, take any $s \in S$ with $(\mathcal{K}, s) \models \Box A$.

We have to prove that $(\mathcal{K}, s) \models \Box\Box A$, i.e. we have to show that sRt implies $(\mathcal{K}, t) \models \Box A$, or, in other words,

$$sRt \text{ implies } (tRu \text{ implies } (\mathcal{K}, u) \models A).$$

Suppose sRt. If tRu, we have sRu by transitivity, so $(\mathcal{K}, u) \models A$ since $(\mathcal{K}, s) \models \Box A$ by hypothesis.

The other cases are left as exercises.

Correspondence theorems

Theorem. If a frame $\mathcal{F} = (S, R)$ validates any one of the schemata 1-10, then R satisfies the corresponding property.

Proof. Consider schema 10. To show that R is weakly directed, suppose sRt and sRu.

Let $\mathcal{K}$ be any model on $\mathcal{F}$ in which $I(p)(v) = 1$ iff uRv.

Then by definition, uRv implies $(\mathcal{K}, v) \models p$, so $(\mathcal{K}, u) \models \Box p$, and hence, as sRu, $(\mathcal{K}, s) \models \Diamond\Box p$. But then as schema 10 is valid in $\mathcal{F}$, $(\mathcal{K}, s) \models \Box\Diamond p$, so as sRt, $(\mathcal{K}, t) \models \Diamond p$. This implies that there exists a v with tRv and $(\mathcal{K}, v) \models p$, i.e. $V(p)(v) = 1$, so uRv; as desired.

Correspondence theorems

Theorem. If a frame $\mathcal{F} = (S, R)$ validates any one of the schemata 1-10, then R satisfies the corresponding property.

Proof. Consider now schema 8. Suppose sRt. Let $\mathcal{K}$ be a Kripke model on $\mathcal{F}$ with $I(p)(v) = 1$ iff $t \neq v$.

Then $(\mathcal{K}, t) \not\models p$, so $(\mathcal{K}, s) \not\models \Box p$.

Hence by validity of schema 8, $(\mathcal{K}, s) \not\models \Box\Box p$, so there exists a u with sRu and $(\mathcal{K}, u) \not\models \Box p$.

Then for some v, uRv and $(\mathcal{K}, v) \not\models p$, i.e. $v = t$, so that uRt, as needed to show that R is weakly dense.

A general result

Property of R:

$$C(m,n,j,k): \quad \forall s_1 \forall s_2 \forall s_3 ((R^m(s_1,s_2) \wedge R^j(s_1,s_3) \to \exists s_4 (R^n(s_2,s_4) \wedge R^k(s_3,s_4)))$$

A general result

Property of R:

$$C(m,n,j,k): \quad \forall s_1 \forall s_2 \forall s_3 ((R^m(s_1,s_2) \wedge R^j(s_1,s_3) \rightarrow \exists s_4 (R^n(s_2,s_4) \wedge R^k(s_3,s_4)))$$

where $R^0(x,y) := x = y$

$$R^1(x,y) := R(x,y)$$

$$R^2(x,y) = \exists u (R(x,u) \wedge R(u,y))$$

$$R^m(x,y) = \exists u_1 \ldots u_{m-1} (R(x,u_1) \wedge \cdots \wedge R(u_{m-1},y))$$

A general result

Theorem. For every $m, n, j, k \in \mathbb{N}$, the axiom

$$\Diamond^m \Box^n P \to \Box^j \Diamond^k P$$

characterizes the class of all frames in which

$$C(m, n, j, k) : \forall s_1 \forall s_2 \forall s_3 ((R^m(s_1, s_2) \wedge R^j(s_1, s_3) \to \exists s_4 (R^n(s_2, s_4) \wedge R^k(s_3, s_4)))$$

is true.

We use the abbreviations

$$\Box^n P = \underbrace{\Box \ldots \Box}_{n \text{ times}} P$$

$$\Diamond^n P = \underbrace{\Diamond \ldots \Diamond}_{n \text{ times}} P$$

A general result

Theorem. For every $m, n, j, k \in \mathbb{N}$, the axiom

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is true.

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In particular, $\Box^0 P$ and $\Diamond^0 P$ stand for P

A general result

Theorem. For every $m, n, j, k \in \mathbb{N}$, the axiom $\Diamond^m \Box^n P \rightarrow \Box^j \Diamond^k P$ characterizes the class of all frames in which $C(m, n, j, k)$ is true, where:

$$C(m, n, j, k) : \forall s_1 \forall s_2 \forall s_3 ((R^m(s_1, s_2) \wedge R^j(s_1, s_3) \rightarrow \exists s_4 (R^n(s_2, s_4) \wedge R^k(s_3, s_4)))$$

Proof "$\Rightarrow$" Let (S, R) be s.t. for every I $(S, R, I) \models \Diamond^m \Box^n P \rightarrow \Box^j \Diamond^k P$. We show that R has property $C(m, n, j, k)$.

Let $s_1, s_2, s_3 \in S$ be such that $R^m(s_1, s_2) \wedge R^j(s_1, s_3)$.

Let I with $I(w, P) = 1$ if $R^n(s_2, w)$ and $I(w, P) = 0$ otherwise.

Then, for $\mathcal{K} = (S, R, I)$ we have $(\mathcal{K}, s_2) \models \Box^n P$, hence $(\mathcal{K}, s_1) \models \Diamond^m \Box^n P$.

Then, by assumption, $(\mathcal{K}, s_1) \models \Box^j \Diamond^k P$.
Since $R^j(s_1, s_3)$, it follows that there exists $s \in S$ such that $R^k(s_3, s)$ and $I(s, P) = 1$, hence by the definition of I, $R^n(s_2, s)$.

A general result

Theorem. For every $m, n, j, k \in \mathbb{N}$, the axiom $\Diamond^m \Box^n P \rightarrow \Box^j \Diamond^k P$ characterizes the class of all frames in which $C(m, n, j, k)$ is true, where:

$C(m, n, j, k) : \forall s_1 \forall s_2 \forall s_3((R^m(s_1, s_2) \wedge R^j(s_1, s_3) \rightarrow \exists s_4(R^n(s_2, s_4) \wedge R^k(s_3, s_4)))$

Proof "$\Leftarrow$" Assume $R \subseteq S \times S$ has the property $C(m, n, j, k)$.
Let $\mathcal{K} = (S, R, I)$ and $s_1 \in S$. We show that $(\mathcal{K}, s_1) \models \Diamond^m \Box^n P \rightarrow \Box^j \Diamond^k P$.

Assume that $(\mathcal{K}, s_1) \models \Diamond^m \Box^n P$.
Then there exists $s_2 \in S$ such that $R^m(s_1, s_2)$ and $(\mathcal{K}, s_2) \models \Box^n P$.

We want to show that $(\mathcal{K}, s_1) \models \Box^j \Diamond^k P$. Let $s_3 \in S$ be such that $R^j(s_1, s_3)$.
Since we assumed that R has property $C(m, n, j, k)$, there exists $s_4 \in S$ such that $R^n(s_2, s_4) \wedge R^k(s_3, s_4)$.

From $R^n(s_2, s_4)$ and $(\mathcal{K}, s_2) \models \Box^n P$ we infer that $I(P, s_4) = 1$.

From this and the fact that $R^k(s_3, s_4)$ it follows that $(\mathcal{K}, s_3) \models \Diamond^k P$.
It follows therefore that $(\mathcal{K}, s_1) \models \Box^j \Diamond^k P$. QED

Exercise

(1) Complete the proofs of the correspondence theorems.

(2) Give a property of R that is necessary and sufficient for $\mathcal{F}$ to validate the schema $A \to \Box A$. Do the same for $\Box \perp$.